PHYS572: Lecture Notes
Fundamentals of Quantum Optics

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Chapter 1

Introduction

The field of *quantum optics* concerns the study of quantum mechanical coherence associated with electromagnetic fields, as well as the interaction of electromagnetic radiation with atoms and molecules. This area has been enormously active in the past few decades since the development of the laser. The field has grown so much that it is not possible to cover the entire field in a single course. The purpose of PHYS 572 is to give an introduction so that you know the basics and can read the literature of the current research.

Motivations for the work in this area has been provided both by the chance to do fundamental physics, and by the possibility of applications such as optical memories, optical communications, etc. Many fundamental experiments leading up to the development of quantum mechanics, including the photoelectric effect, blackbody radiation were all atom-optical experiments. Many of the *gedanken* experiments that were beyond the technology of the day, but whose theoretical study was crucial to the development of quantum theory, are now realizable in the laboratory. Quantum optics therefore offers the opportunity to gain new insight into these features of quantum theory by actually carrying out in the laboratory the *gedanken* experiments.

First let us examine the concept of photon.

1.1 The Photon Concept

The wave-particle duality is one of the most important concepts in quantum mechanics. In quantum optics, we are particularly interested in the wave-particle nature of the light. Classically, the wave theory describes light perfectly well: light is electromagnetic waves obeying the Maxwell's equations. So what kinds of phenomena demand to invoke the particle nature of light for explanation?

Planck's blackbody radiation theory prompted the quantum revolution. But he was thinking in terms of quantizing the energies of the oscillators in the walls of his cavity, not quantizing the radiation field itself. Einstein was granted the Nobel prize for his explanation of photoelectric effect using the photon hypothesis. But a semiclassical theory, in which the light is treated classically but the matter is treated quantum mechanically according to the Schrödinger equation, explains the effect equally well. We need to look elsewhere.

Now consider the light produced in the "atomic cascade" of Ca atom spontaneously emitting through

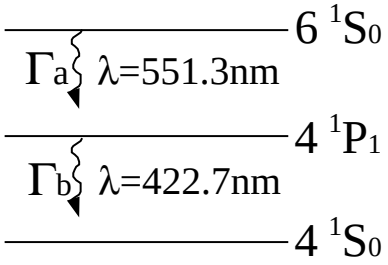

Figure 1-1: Level diagram of Ca.

the transitions $6\,^1S_0 \rightarrow 4\,^1P_1 \rightarrow 4\,^1S_0$, whose level diagram is shown in Fig. 1. The intermediate $4\,^1P_1$ state is very short lived, i.e., $\Gamma_b \gg \Gamma_a$. Thus any photon emitted at 551.3 nm on the upper transition must be correlated with the emission of a 422.7 nm photon on the lower transition. In addition, because the atom starts and ends in a state with zero total angular momentum, these two photons must be anti-correlated in their helicity. Are there observable phenomena associated with the light produced in such a system that cannot be accounted for the by the semiclassical theory? The answer is, yes, many!

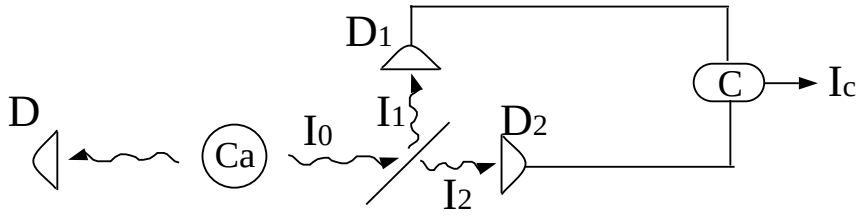

Figure 1-2: Light detection experiment.

Here is one particular striking example. Consider the experiment sketched in Fig. 2. Using the photons produced in the atomic cascade of Ca, an ensemble of single photon states is prepared by using one of the photons (falling to detector 'D') as a "trigger" for coincidence detection of the other. A 50:50 beam splitter and two photo detectors D_1 and D_2 are placed equidistant from the Ca source. Let I_0 be the intensity of the incident light. According to the classical theory, the incident field, no matter how weak, will be equally divided at the beam splitter. Hence we have $I_1 = I_2 = I_0/2$. The coincidence counter C measures the intensity correlation $I_c = \langle I_1 I_2 \rangle = \langle I_0^2 \rangle / 4$. Using the inequality $\langle I_0^2 \rangle \geq \langle I_0 \rangle^2$, we have then

$$I_c = \langle I_1 I_2 \rangle \geq \langle I_1 \rangle \langle I_2 \rangle \tag{1.1}$$

This result is in sharp contrast to the prediction of quantum mechanics. Suppose the radiation incidence on the beam splitter consists of a single photon, and the experiment is repeated many times with an ensemble of identically prepared photons. Because the photon is an indivisible object, it cannot be detected at both output ports of the beam splitter simultaneously. They we expect the number of coincidence counts to be strictly zero.

The experiment was carried out by Aspect *et al.* at the Institut d'Optique in Orsay, France [J. Opitcs (Paris), **20**, 119 (1989)]. Their result was consistent with the quantum prediction, hence experimentally confirmed the existence of photon.

One may ask why use atomic cascade? One can attenuate light from a classical light source to such a degree that the average number of photons detected in a time interval T is much smaller than one, then the number of coincidence counts should be expected to violate the classical inequality (1.1). Aspect *et al.* performed this experiment by attenuating a light pulse produced by a photodiode such that in the duration of one pulse the photon number is ~ 0.01. However, the inequality (1.1) is not violated under this situation.

1.2 Resonant Photon-Atom Interaction

The second part of this course concerns the interaction between photon and atom. *We assume that the atom and the field are separate entities that interact with each other while maintaining their separate identities.* Thus the field is neither too strong to yank the atom apart, nor too high in frequency to directly ionize the atom to a highly excited state.

We can use the Bohr-Sommerfeld model to estimate the range of parameters we are interested in. The simplest Bohr atom has a proton at the center and an electron orbiting it with Bohr radius $a_0 = 0.529$ angstrom. The Coulomb interaction gives rise to a electric field about 5.14×10^{11} V/m. To achieve such an electric field, requires a laser with intensity 8.3×10^{16} W/cm^2. So quantum optics as we defined does not exist for fields stronger than about 10^{17} W/cm^2. At this point the electron interacts as strongly with optical field as it does the nucleus.

In fact it is not necessary to achieve fields remotely this strong to produce dramatic effects. The cross section for an atom to absorb an optical photon is generally of the order of one squared angstrom, but when the frequency of the photon matches the transition frequency of the atom, the cross section can be enhanced by many orders of magnitude (to be as large as the square of the optical wavelength). An isolated atom can dissipate the absorbed energy only by reradiating it. The maximum rate at which an atom can radiate is the spontaneous emission rate. For a typical optical transition that rate is about 10^8 photons/second. A resonant laser beam with an intensity of 1 mW/cm^2 can supply photons this fast to the atom. It is therefore this greatly enhanced interaction of resonant optical fields that will be the main subject of this course.

Chapter 2

Quantum Mechanical Background

2.1 Review of Quantum Mechanics

- States: Our best possible knowledge about a quantum mechanical system, represented by wave function $\psi(\mathbf{r}, t)$ which allows us to calculate the expectation values of all observables of interest, the quantity $|\psi(\mathbf{r}, t)|^2 \, d^3r$ is the probability of finding the system in the volume element d^3r.
- Observables: Physical quantities, represented by Hermitian operators $\hat{O}(\mathbf{r})$ and its expectation value is given in terms of $\psi(\mathbf{r}, t)$ by $\langle\psi|\hat{O}|\psi\rangle(t) = \int d^3r \, \psi^*(\mathbf{r}, t) O(\mathbf{r}) \psi(\mathbf{r}, t)$.
- Dynamics: Time evolution of the quantum system. Different pictures: Schrödinger, Heisenberg and Interaction.

2.2 Dynamics: Schrödinger, Heisenberg and Interaction Pictures

Different pictures may offer different insights to a particular problem. One should use the picture that makes life easier.

In the Schrödinger picture, the state wave function evolves in time while the operators are time-independent. Integrating the Schrödinger equation results in the dynamics of the wave function:

$$|\psi_S(t)\rangle = U(t, 0)|\psi_S(0)\rangle$$

where $U(t, 0)$ is the time evolution operator which satisfies

$$i\hbar \frac{\partial}{\partial t} U(t, 0) = H U(t, 0)$$

If H is time-independent, then $U(t, 0) = \exp(-iHt/\hbar)$. The expectation value of an operator $\hat{O}^{(S)}$ is then

$$\langle \hat{O}^{(S)} \rangle(t) = \langle \psi_S(t)|\hat{O}^{(S)}|\psi_S(t)\rangle = \langle \psi_S(0)|U^{-1}(t, 0)\hat{O}^{(S)}U(t, 0)|\psi_S(0)\rangle$$

In the Heisenberg picture, the state is time-independent (i.e., $|\psi_H\rangle = |\psi_S(0)\rangle$), while the operator evolves in time

$$\hat{O}^{(H)}(t) = U^{-1}(t, 0)\hat{O}^{(S)}U(t, 0)$$

It's easy to see that $\hat{O}^{(H)}(t)$ satisfies the Heisenbergy equation of motion:

$$i\hbar\frac{d}{dt}\hat{O}^{(H)}(t) = [\hat{O}^{(H)}(t), H]$$

In many cases, we find that we are dealing with a Hamiltonian of the form $H = H_0 + V$, where H_0 is the "bare" Hamiltonian and V represents the interaction. Usually we already know the solutions of the problem with the bare Hamiltonian H_0. In these situations, the Interaction picture is often the most convenient one. More formally, we substitute the state vector

$$|\psi_S(t)\rangle = U_0(t,0)|\psi_I(t)\rangle$$

where $U_0(t,0) = \exp(-iH_0t/\hbar)$, into the Schrödinger equation. We then have

$$i\hbar\frac{d}{dt}|\psi_I(t)\rangle = V_I(t)|\psi_I(t)\rangle$$

with the Interaction picture interaction energy

$$V_I(t) = U_0^{-1}(t,0)VU_0(t,0)$$

We can also find the expectation value of an operator as

$$\langle\hat{O}\rangle(t) = \langle\psi_I(t)|\hat{O}^{(I)}(t)|\psi_I(t)\rangle$$

where

$$\hat{O}^{(I)}(t) = U_0^{-1}(t,0)\hat{O}^{(S)}U_0(t,0)$$

Therefore we eliminate the part of the problem whose solution we already know, and concentrate on the unknown part.

2.3 Density Operator

2.3.1 A state vector is not enough

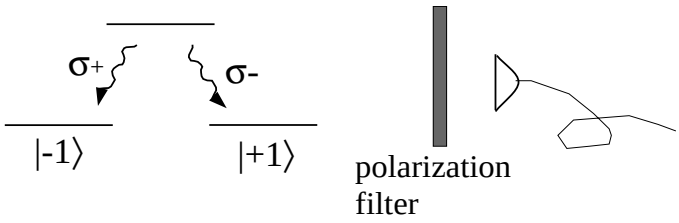

Figure 2-1: Light detection experiment.

Consider the experiment outlined in the figure. The atom is initially prepared in the excited state. It can decay to state $|-1\rangle$ ($|+1\rangle$) by emitting a σ_+-polarized (σ_--polarized) photon. A photodector is placed after a polarization filter. Suppose the experimentalist told us that she used a curcular-polarized filter, but refused

to tell us whether it's σ_+ or σ_- polarized, i.e., she had a 50:50 chance to choose either one. After a photon is detected, what is the state of the atom? We can not say the state is given by a quantum superposition of $|-1\rangle$ and $|+1\rangle$, since the uncertainty here is 'classical'.

A quantum state can be described by a state vector. However, in many situations, we don't have a full knowledge about the state of the system. Such situations arise, for example, when the system is coupled to a reservoir and we can no longer keep track of all the degrees of freedom.

Let us try to gain some insight from the following example. Given a particle with a state described by state vector $|\psi\rangle$. The probability density to find the particle at x is

$$P(x) = |\langle x|\psi\rangle|^2 = \langle x|\psi\rangle\langle\psi|x\rangle = \langle x|\hat{\rho}|x\rangle$$

where we have introduced the hermitian operator

$$\hat{\rho} \equiv |\psi\rangle\langle\psi|$$

This operator is called the **density operator** since we can use it to calculate probability densities.

Suppose the state $|\psi\rangle$ is expanded onto a complete basis $\{|m\rangle\}$ as

$$|\psi\rangle = \sum_m c_m|m\rangle$$

then the density operator reads:

$$\hat{\rho} = \sum_{m,n} c_m c_n^*|m\rangle\langle n| = \sum_{m,n} \rho_{mn}|m\rangle\langle n|$$

The complex-valued numbers $\rho_{mn} = c_m c_n^*$ form a matrix consisting of products made out of the expansion coefficients c_m. The matrix formed by ρ_{mn} is called the **density matrix**.

Now suppose we don't have good knowledge about the state. We only know that the system has probability $P_m = |c_m|^2$ to be in state $|m\rangle$, but no information on phase is gained. In other words, we have

$$c_m = \sqrt{P_m}\, e^{i\phi_m}$$

where ϕ_m is a random phase. Then we need to average over these phases in order to calculate any expectation values. The density matrix element ρ_{mn} averaged over the phase becomes

$$\rho_{mn} = \overline{c_m c_n^*} = \sqrt{P_m P_n}\overline{e^{i(\phi_m - \phi_n)}} = P_m \delta_{mn}$$

and the density operator becomes

$$\hat{\rho} = \sum_m P_m|m\rangle\langle m|$$

In conclusion, we have two kinds of averages: The first one results from quantum mechanics and the fact that a quantum state can only provide a statistical description. The second average is a classical one. It reflects the fact that we do not have complete information about the system (in the example above, we don't know the phases of the probability amplitudes): We do not know in which quantum state the system is.

In the example given above, we have a 50:50 mixture of the states $|\pm\rangle$ which can be described by the density operator: $\hat{\rho} = (|-1\rangle\langle -1| + |+1\rangle\langle +1|)/2$.

(Now think the following question: what if the experimentalist told us that she used a linear polarization filter? What if we know the atom has emitted a photon, but the photon is left undetected?)

2.3.2 Definition and properties

For a set of states $|\psi_m\rangle$ ($m = 0, 1, 2, ...$), the density operator is defined as

$$\hat{\rho} = \sum_m P_m |\psi_m\rangle\langle\psi_m|$$

where P_m is the classical probability with which state $|\psi_m\rangle$ appears. Note that in the definition, the states $|\psi_m\rangle$ don't have to form an orthonormal set, but the density operator is most conveniently defined if they do. So we'll take such an assumption. Under this condition, we have

$$\langle\psi_m|\hat{\rho}|\psi_m\rangle = P_m$$

Since P_m are probabilities, they have to add up to unity, hence

$$\mathrm{Tr}\hat{\rho} \equiv \sum_m \langle\psi_m|\hat{\rho}|\psi_m\rangle = \sum_m P_m = 1$$

Let us take this opportunity to say a few words about the trace of operator.

- The definition of the trace of the operator $\hat{O}$ reads

  $$\mathrm{Tr}\hat{O} \equiv \sum_m \langle\psi_m|\hat{O}|\psi_m\rangle$$

  where $|\psi_m\rangle$ is a complete set of states. Hence

  $$\mathbf{1} = \sum_m |\psi_m\rangle\langle\psi_m|$$

  Therefore,

  $$\hat{O} = \mathbf{1}\hat{O}\mathbf{1} = \sum_{mn} |\psi_m\rangle\langle\psi_m|\hat{O}|\psi_n\rangle\langle\psi_n| = \sum_{mn} O_{mn}|\psi_m\rangle\langle\psi_n|$$

  with $O_{mn} = \langle\psi_m|\hat{O}|\psi_n\rangle$. And the trace of $\hat{O}$ is indeed the sum over the diagonal elements O_{mm}.

- **Trace is independent of representation** To show this, let us consider a different complete set of orthonormal states $|\phi_n\rangle$ and $\mathbf{1} = \sum_n |\phi_n\rangle\langle\phi_n|$. Then

  $$\begin{aligned}
  \mathrm{Tr}\hat{O} &= \sum_m \langle\psi_m|\mathbf{1}\hat{O}\mathbf{1}|\psi_m\rangle = \sum_{m,n,l} \langle\psi_m|\phi_n\rangle\langle\phi_n|\hat{O}|\phi_l\rangle\langle\phi_l|\psi_m\rangle \\
  &= \sum_{n,l} \langle\phi_l \left(\sum_m |\psi_m\rangle\langle\psi_m|\right) \phi_n\rangle\langle\phi_n|\hat{O}|\phi_l\rangle \\
  &= \sum_{n,l} \delta_{nl}\langle\phi_n|\hat{O}|\phi_l\rangle \\
  &= \sum_n \langle\phi_n|\hat{O}|\phi_n\rangle
  \end{aligned}$$

- $\mathrm{Tr}[\hat{A}\hat{B}] = \mathrm{Tr}[\hat{B}\hat{A}]$

- **Expectation value is the Trace** The trace operation allows us to calculate expectation values of operators. First, the expectation value of operator $\hat{O}$ is defined as

  $$\langle\hat{O}\rangle = \sum_m P_m \langle\psi_m|\hat{O}|\psi_m\rangle$$

It can be easily shown that

$$\langle \hat{O} \rangle = \mathrm{Tr}[\hat{O}\hat{\rho}]$$

Using the above property, we have the following properties for density operator:

- $\langle \hat{\rho} \rangle \leq 1$ since $\langle \hat{\rho} \rangle = \mathrm{Tr}[\hat{\rho}\hat{\rho}] = \sum_m P_m^2 \leq \sum_m P_m = 1$. The equal sign occurs if and only if there is one single state $|\psi_l\rangle$ contributing to the sum, i.e., $P_m = \delta_{lm}$. In this case, the quantum system is described by a single state, and is therefore a **pure state**. Otherwise, the state is call a **mixed state**.

- The matrix elements of the density matrix in any basis $\{|\psi_n\rangle\}$ is given by

  $$\rho_{nm} = \langle \psi_n | \hat{\rho} | \psi_m \rangle = \rho_{mn}^*$$

  and are constrained by the inequality

  $$\rho_{nm}\rho_{mn} = |\rho_{nm}|^2 \leq \rho_{nn}\rho_{mm} \tag{2.1}$$

  where the equal sign holds for pure state.

The proof of the inequality goes as follows: Define two states as

$$|\phi_1\rangle = \hat{\rho}^{1/2}|\psi_n\rangle, \quad |\phi\rangle = \hat{\rho}^{1/2}|\psi_m\rangle$$

then we have

$$\langle \phi_1 | \phi_2 \rangle = \rho_{nm}, \quad \langle \phi_1 | \phi_1 \rangle = \rho_{nn}, \quad \langle \phi_2 | \phi_2 \rangle = \rho_{mm}$$

and the inequality (3.1) follows from the Cauchy-Schwarz inequality

$$|\langle \phi_1 | \phi_2 \rangle|^2 \leq \langle \phi_1 | \phi_1 \rangle \langle \phi_2 | \phi_2 \rangle$$

To show that the equal sign holds in (3.1) for a pure state, we realize that the operator $\hat{\rho}^{1/2}$ can be written as

$$\hat{\rho}^{1/2} = \sum_n \sqrt{\sigma_n} |\lambda_n\rangle\langle\lambda_n|$$

where $\{|\lambda_n\rangle\}$ is a set of basis states where the density matrix is diagonal, i.e., $\hat{\rho} = \sum_n \sigma_n |\lambda_n\rangle\langle\lambda_n|$. For a pure state, one and only one of the σ_n's will be equal to 1 and all the other will be zero. Let's say $\sigma_n = \delta_{n1}$, then we have

$$|\phi_1\rangle = \langle \lambda_1 | \psi_n \rangle |\lambda_1\rangle, \quad |\phi_2\rangle = \langle \lambda_1 | \psi_m \rangle |\lambda_1\rangle$$

i.e., $|\phi_1\rangle$ and $|\phi_2\rangle$ correspond to the same state with a c-number constant factor. Thus, the equal sign holds for the Cauchy-Schwarz inequality, hence also for (3.1).

2.3.3 Reduced Density Operator

A system may consist of two subsystems A and B, and the total density operator is given by $\hat{\rho}_{AB}$. If we are only care about the subsystem A, for example, we want to calculate the expectation value of an operator $\hat{O}_A$ that only operates on A, then we have

$$\langle \hat{O} \rangle = \mathrm{Tr}_{AB}[\hat{O}_A \hat{\rho}_{AB}] = \mathrm{Tr}_A[\hat{O}_A \mathrm{Tr}_B[\hat{\rho}_{AB}]] = \mathrm{Tr}_A[\hat{O}_A \hat{\rho}_A]$$

where $\hat{\rho}_A = \mathrm{Tr}_B[\hat{\rho}_{AB}]$ is call the *reduced density operator* for subsystem A.

The state is **separable** if $\hat{\rho}_{AB} = \hat{\rho}_A \otimes \hat{\rho}_B$, and **entangled** if otherwise. In the light detection example above, if the photon is left on detected, we have an entangled state $|\psi\rangle = (|+1\rangle|\sigma_-\rangle + |-1\rangle|\sigma_+\rangle)/\sqrt{2}$ with the density operator

$$\hat{\rho} = \frac{1}{2}\left[|+1\rangle\langle+1| \otimes |\sigma_-\rangle\langle\sigma_-| + |+1\rangle\langle-1| \otimes |\sigma_-\rangle\langle\sigma_+| + |-1\rangle\langle+1| \otimes |\sigma_+\rangle\langle\sigma_-| + |-1\rangle\langle-1| \otimes |\sigma_+\rangle\langle\sigma_+|\right]$$

2.3.4 Time evolution of the density operator

More generally, the density matrix operator, introduced as a device to represent our knowledge of the initial state of the quantum system under study, is frequently employed for time-dependent purposes. How can a bookkeeping statement about the system at $t = 0$ acquire dynamic properties? The answer is, by switching from the Heisenberg picture to the Schrödinger picture. To see this, consider the expectation value of $\hat{O}$:

$$\langle\hat{O}\rangle(t) = \mathrm{Tr}[\hat{\rho}\hat{O}^{(H)}(t)] = \mathrm{Tr}[\hat{\rho}U^{-1}(t,0)\hat{O}^{(S)}U(t,0)] = \mathrm{Tr}[U(t,0)\hat{\rho}U^{-1}(t,0)\hat{O}^{(S)}] = \mathrm{Tr}[\hat{\rho}(t)\hat{O}^{(S)}]$$

where

$$\hat{\rho}(t) \equiv U(t,0)\hat{\rho}U^{-1}(t,0)$$

Note that $\hat{\rho}(t)$ and $\hat{\rho}$ are **not** related in the same way as any Heisenberg operators $\hat{O}^{(H)}(t)$ and $\hat{O}^{(H)}(0) = \hat{O}^{(S)}$ are related, since

$$\hat{O}^{(H)}(t) = U^{-1}(t,0)\hat{O}^{(S)}U(t,0)$$

The consequence of this difference is that $\hat{\rho}(t)$ obeys an equation of motion:

$$i\hbar\frac{d}{dt}\hat{\rho}(t) = [H, \hat{\rho}(t)]$$

which is sometimes called the quantum Liouville equation.

In the Interaction picture, we have

$$\hat{\rho}^{(I)}(t) = U_0^{-1}(t,0)\hat{\rho}(t)U_0(t,0)$$

and the equation of motion takes the form

$$i\hbar\frac{d}{dt}\hat{\rho}^{(I)}(t) = [V_I(t), \hat{\rho}^{(I)}(t)] \tag{2.2}$$

which can be formally integrated to give

$$\hat{\rho}^{(I)}(t) = \hat{\rho}^{(I)}(0) + \frac{1}{i\hbar}\int_0^t [V_I(t_1), \hat{\rho}^{(I)}(t_1)]dt_1 \tag{2.3}$$

To first order, we can substitute $\hat{\rho}^{(I)}(t_1)$ in the integral by its initial value, then

$$\hat{\rho}^{(I)}(t) = \hat{\rho}^{(I)}(0) + \frac{1}{i\hbar}\int_0^t [V_I(t_1), \hat{\rho}^{(I)}(0)]dt_1$$

Similarly we can obtain the second order approximation as

$$\hat{\rho}^{(I)}(t) = \hat{\rho}^{(I)}(0) + \frac{1}{i\hbar}\int_0^t dt_1\,[V_I(t_1), \hat{\rho}^{(I)}(0)] + \frac{1}{(i\hbar)^2}\int_0^t dt_1\int_0^{t_1} dt_2\,[V_I(t_1), [V_I(t_2), \hat{\rho}(0)]]$$

Chapter 3

Density Matrix

3.1 A state vector is not enough

A quantum state can be described by a state vector. However, in many situations, we don't have a full knowledge about the state of the system. Such situations arise, for example, when the system is coupled to a reservoir and we can no longer keep track of all the degrees of freedom.

Let us try to gain some insight from the following example. Given a particle with a state described by state vector $|\psi\rangle$. The probability density to find the particle at x is

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Suppose the state $|\psi\rangle$ is expanded onto a complete basis $\{|m\rangle\}$ as

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then the density operator reads:

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3.2 Definition and properties

For a set of states $|\psi_m\rangle$ ($m = 0, 1, 2, ...$), the density operator is defined as

$$\hat{\rho} = \sum_m P_m |\psi_m\rangle\langle\psi_m|$$

where P_m is the classical probability with which state $|\psi_m\rangle$ appears. Note that in the definition, the states $|\psi_m\rangle$ don't have to form an orthonormal set, but the density operator is most conveniently defined if they do. So we'll take such an assumption. Under this condition, we have

$$\langle\psi_m|\hat{\rho}|\psi_m\rangle = P_m$$

Since P_m are probabilities, they have to add up to unity, hence

$$\mathrm{Tr}\hat{\rho} \equiv \sum_m \langle\psi_m|\hat{\rho}|\psi_m\rangle = \sum_m P_m = 1$$

Let us take this opportunity to say a few words about the trace of operator.

- The definition of the trace of the operator $\hat{O}$ reads

  $$\mathrm{Tr}\hat{O} \equiv \sum_m \langle\psi_m|\hat{O}|\psi_m\rangle$$

  where $|\psi_m\rangle$ is a complete set of states. Hence

  $$\mathbf{1} = \sum_m |\psi_m\rangle\langle\psi_m|$$

  Therefore,

  $$\hat{O} = \mathbf{1}\hat{O}\mathbf{1} = \sum_{mn} |\psi_m\rangle\langle\psi_m|\hat{O}|\psi_n\rangle\langle\psi_n| = \sum_{mn} O_{mn}|\psi_m\rangle\langle\psi_n|$$

  with $O_{mn} = \langle\psi_m|\hat{O}|\psi_n\rangle$. And the trace of $\hat{O}$ is indeed the sum over the diagonal elements O_{mm}.

- **Trace is independent of representation** To show this, let us consider a different complete set of

orthonormal states $|\phi_n\rangle$ and $\mathbf{1} = \sum_n |\phi_n\rangle\langle\phi_n|$. Then

$$\begin{aligned}
\mathrm{Tr}\hat{O} &= \sum_m \langle\psi_m|\mathbf{1}\hat{O}\mathbf{1}|\psi_m\rangle = \sum_{m,n,l} \langle\psi_m|\phi_n\rangle\langle\phi_n|\hat{O}|\phi_l\rangle\langle\phi_l|\psi_m\rangle \\
&= \sum_{n,l} \langle\phi_l \left(\sum_m |\psi_m\rangle\langle\psi_m|\right) \phi_n\rangle\langle\phi_n|\hat{O}|\phi_l\rangle \\
&= \sum_{n,l} \delta_{nl}\langle\phi_n|\hat{O}|\phi_l\rangle \\
&= \sum_n \langle\phi_n|\hat{O}|\phi_n\rangle
\end{aligned}$$

- $\mathrm{Tr}[\hat{A}\hat{B}] = \mathrm{Tr}[\hat{B}\hat{A}]$
- **Expectation value is the Trace** The trace operation allows us to calculate expectation values of operators. First, the expectation value of operator $\hat{O}$ is defined as

  $$\langle\hat{O}\rangle = \sum_m P_m\langle\psi_m|\hat{O}|\psi_m\rangle$$

  It can be easily shown that

  $$\langle\hat{O}\rangle = \mathrm{Tr}[\hat{O}\hat{\rho}]$$

Using the above property, we have the following properties for density operator:

- $\langle\hat{\rho}\rangle \leq 1$ since $\langle\hat{\rho}\rangle = \mathrm{Tr}[\hat{\rho}\hat{\rho}] = \sum_m P_m^2 \leq \sum_m P_m = 1$. The equal sign occurs if and only if there is one single state $|\psi_l\rangle$ contributing to the sum, i.e., $P_m = \delta_{lm}$. In this case, the quantum system is described by a single state, and is therefore a **pure state**. Otherwise, the state is call a **mixed state**.
- The matrix elements of the density matrix in any basis $\{|\psi_n\rangle\}$ is given by

  $$\rho_{nm} = \langle\psi_n|\hat{\rho}|\psi_m\rangle = \rho_{mn}^*$$

  and are constrained by the inequality

  $$\rho_{nm}\rho_{mn} \leq \rho_{nn}\rho_{mm} \tag{3.1}$$

  where the equal sign holds for pure state.

The proof of the inequality goes as follows: Define two states as

$$|\phi_1\rangle = \hat{\rho}^{1/2}|\psi_n\rangle, \quad |\phi\rangle = \hat{\rho}^{1/2}|\psi_m\rangle$$

then we have

$$\langle\phi_1|\phi_2\rangle = \rho_{nm}, \quad \langle\phi_1|\phi_1\rangle = \rho_{nn}, \quad \langle\phi_2|\phi_2\rangle = \rho_{mm}$$

and the inequality (3.1) follows from the Cauchy-Schwarz inequality

$$|\langle\phi_1|\phi_2\rangle|^2 \leq \langle\phi_1|\phi_1\rangle\langle\phi_2|\phi_2\rangle$$

To show that the equal sign holds in (3.1) for a pure state, we realize that the operator $\hat{\rho}^{1/2}$ can be written as

$$\hat{\rho}^{1/2} = \sum_n \sqrt{\sigma_n} |\lambda_n\rangle\langle\lambda_n|$$

where $\{|\lambda_n\rangle\}$ is a set of basis states where the density matrix is diagonal, i.e., $\hat{\rho} = \sum_n \sigma_n |\lambda_n\rangle\langle\lambda_n|$. For a pure state, one and only one of the σ_n's will be equal to 1 and all the other will be zero. Let's say $\sigma_n = \delta_{n1}$, then we have

$$|\phi_1\rangle = \langle\lambda_1|\psi_n\rangle|\lambda_1\rangle, \quad |\phi_2\rangle = \langle\lambda_1|\psi_m\rangle|\lambda_1\rangle$$

i.e., $|\phi_1\rangle$ and $|\phi_2\rangle$ correspond to the same state with a c-number constant factor. Thus, the equal sign holds for the Cauchy-Schwarz inequality, hence also for (3.1).

3.3 Time evolution of the density operator

The density matrix operator, introduced as a device to represent our knowledge of the initial state of the quantum system under study, is frequently employed for time-dependent purposes. How can a bookkeeping statement about the system at $t = 0$ acquire dynamic properties? The answer is, by switching from the Heisenberg picture to the Schrödinger picture. To see this, consider the expectation value of $\hat{O}$:

$$\langle\hat{O}(t)\rangle = \mathrm{Tr}[\hat{\rho}\hat{O}] = \mathrm{Tr}[\hat{\rho}U^{-1}(t,0)\hat{O}(0)U(t,0)] = \mathrm{Tr}[U(t,0)\hat{\rho}U^{-1}(t,0)\hat{O}(0)] = \mathrm{Tr}[\hat{\rho}(t)\hat{O}(0)]$$

where

$$\hat{\rho}(t) \equiv U(t,0)\hat{\rho}U^{-1}(t,0)$$

and $U(t,0)$ is the time evolution operator which satisfies

$$i\hbar\frac{\partial}{\partial t}U(t,0) = HU(t,0)$$

Note that $\hat{\rho}(t)$ and $\hat{\rho}$ are **not** related in the same way as any Heisenberg operators $\hat{O}(t)$ and $\hat{O}(0)$ are related, since

$$\hat{O}(t) = U^{-1}(t,0)\hat{O}(0)U(t,0)$$

The consequence of this difference is that $\hat{\rho}(t)$ obeys an equation of motion:

$$i\hbar\dot{\hat{\rho}}(t) = [H, \hat{\rho}]$$

which is sometimes called the quantum Liouville equation.

3.4 Application to two-level atom

The Hilbert space is spanned by states $|g\rangle$ and $|e\rangle$. So the density operator can be represented by a 2×2 matrix whose elements are given by $\rho_{ij} = \langle i|\hat{\rho}|j\rangle$ $(i, j = g, e)$. Let us find the relationship between ρ_{ij} and $\hat{\sigma}_{ij}$ we encountered earlier. To do so, let us check that

$$\sigma_{ij}(t) = \langle\hat{\sigma}_{ij}(t)\rangle = \mathrm{Tr}[\hat{\rho}(t)\hat{\sigma}_{ij}(0)] = \mathrm{Tr}[\hat{\rho}(t)|i\rangle\langle j|] = \sum_{k=g,e}\langle k|\hat{\rho}(t)|i\rangle\langle j|k\rangle = \langle j|\hat{\rho}(t)|i\rangle = \rho_{ji}(t)$$

The optical Bloch equations can be also written in terms of ρ_{ij}.

Chapter 4

Quantization of the Electromagnetic Field

4.1 Maxwell's Equations and the Coulomb Gauge

In quantum theory the field vectors must be taken as operators instead of the algebraic quantities of classical theory, but both theories are based on the same Maxwell's equations:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{4.1}$$

$$\frac{1}{\mu_0}\nabla \times \mathbf{B} = \varepsilon_0 \frac{\partial \mathbf{E}}{\partial t} + \mathbf{J} \tag{4.2}$$

$$\varepsilon_0 \nabla \cdot \mathbf{E} = \rho \tag{4.3}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{4.4}$$

where $\mathbf{J}$ and ρ are the current and charge density, respectively.

These equations can be re-expressed in terms of the scalar and vector potentials, ϕ and $\mathbf{A}$, respectively, where

$$\mathbf{B} = \nabla \times \mathbf{A}, \quad \nabla\phi = -\mathbf{E} - \frac{\partial \mathbf{A}}{\partial t}$$

With these definitions, Eqs. (4.1) and (4.4) are automatically satisfied. Putting the definition of the potentials into the remaining two equations, we obtain the field equations

$$\begin{aligned} \nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} + \frac{1}{c^2}\frac{\partial}{\partial t}\nabla\phi + \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} &= \mu_0 \mathbf{J} \\ -\varepsilon_0 \nabla^2 \phi - \varepsilon_0 \nabla \cdot \left(\frac{\partial \mathbf{A}}{\partial t}\right) &= \rho \end{aligned}$$

where $c = 1/\sqrt{\varepsilon_0 \mu_0}$ is the vacuum speed of light.

The field equations can be further simplified if a particular *gauge* is chosen. The reason is that although potentials ϕ and $\mathbf{A}$ uniquely determine the fields $\mathbf{E}$ and $\mathbf{B}$, the reverse is not true: The fields are the same

for pairs of potentials $(\mathbf{A}, \phi)$ and $(\mathbf{A}', \phi')$ related by a *gauge transformation*:

$$\mathbf{A} = \mathbf{A}' - \nabla\Xi, \quad \phi = \phi' + \frac{\partial \Xi}{\partial t} \tag{4.5}$$

where the *gauge function* Ξ is an arbitrary scalar function of the position $\mathbf{r}$ and time t. In quantum optics, we always use the **Coulomb gauge** where the vector potential satisfies the condition

$$\nabla \cdot \mathbf{A} = 0$$

The field equations under the Coulomb gauge becomes

$$-\nabla^2 \mathbf{A} + \frac{1}{c^2}\frac{\partial}{\partial t}\nabla\phi + \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = \mu_0 \mathbf{J} \tag{4.6}$$

$$-\nabla^2 \phi = \rho/\varepsilon_0 \tag{4.7}$$

Hence the scalar potential now satisfies Poisson's equation of electrostatics.

Further simplification can be achieved by separate the fields into transverse and longitudinal components. According to the Helmholtz's theorem, any vector field can be written as a sum of two components, one of which has zero divergence and the other has zero curl. For the current density, we have

$$\mathbf{J} = \mathbf{J}_T + \mathbf{J}_L$$

where $\nabla \cdot \mathbf{J}_T = 0$ and $\nabla \times \mathbf{J}_L = 0$. $\mathbf{J}_T$ is called the *transverse* or *solenoidal* component and $\mathbf{J}_L$ the *longitudinal* or *irrotational* component. With this definition, the vector potential $\mathbf{A}$ under Coulomb gauge and the magnetic field $\mathbf{B}$ are completely transverse. Eq. (4.6) can then separated into its transverse and longitudinal parts as

$$-\nabla^2 \mathbf{A} + \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = \mu_0 \mathbf{J}_T \tag{4.8}$$

$$\frac{1}{c^2}\frac{\partial}{\partial t}\nabla\phi = \mu_0 \mathbf{J}_L \tag{4.9}$$

Hence the vector potential satisfies a wave equation. Eliminating ϕ from (4.7) and (4.9) we have

$$\nabla \cdot \mathbf{J}_L = -\partial\rho/\partial t$$

which is the equation of charge conservation.

The electric field can be similarly decomposed as $\mathbf{E} = \mathbf{E}_T + \mathbf{E}_L$ where

$$\mathbf{E}_T = -\partial \mathbf{A}/\partial t, \quad \mathbf{E}_L = -\nabla\phi$$

The Maxwell's equations can then be separated into a transverse part which are associated with the vector potential

$$\nabla \times \mathbf{E}_T = -\frac{\partial \mathbf{B}}{\partial t} \tag{4.10}$$

$$\frac{1}{\mu_0}\nabla \times \mathbf{B} = \varepsilon_0 \frac{\partial \mathbf{E}_T}{\partial t} + \mathbf{J}_T \tag{4.11}$$

$$\nabla \cdot \mathbf{E}_T = 0 \tag{4.12}$$

$$\nabla \cdot \mathbf{B} = 0 \tag{4.13}$$

and a longitudinal part which are associated with the scalar potential

$$\nabla \cdot \mathbf{E}_L = \rho/\varepsilon \tag{4.14}$$

$$\mathbf{J}_L = -\varepsilon_0 \frac{\partial \mathbf{E}_L}{\partial t} \tag{4.15}$$

Let us further consider the energy of the field

$$H = \frac{1}{2}\int d^3r \left[\varepsilon_0 \mathbf{E}^2 + \frac{1}{\mu_0}\mathbf{B}^2\right] = \frac{1}{2}\int d^3r \left[\varepsilon_0 \left(\mathbf{E}_T^2 + 2\mathbf{E}_T \cdot \mathbf{E}_L + \mathbf{E}_L^2\right) + \frac{1}{\mu_0}\mathbf{B}^2\right]$$

Examine the two terms that contain the longitudinal electric field.

$$\int d^3r\, \mathbf{E}_T \cdot \mathbf{E}_L = -\int d^3r\, \nabla\phi \cdot \mathbf{E}_T = -\int d^3r\, \nabla \cdot (\phi \mathbf{E}_T) = -\oint_\infty d\mathbf{S} \cdot (\phi \mathbf{E}_T) = 0$$

as long as ϕ falls off at least as fast as $1/r$ and $\mathbf{E}_T$ falls off as fast as $1/r^2$ as $r \to \infty$. For the other term we have

$$\int d^3r\, \mathbf{E}_L^2 = \int d^3r\, |\nabla\phi|^2 = \int d^3r\, \left[\nabla \cdot (\phi\nabla\phi) - \phi\nabla^2\phi\right] = \frac{1}{\varepsilon_0}\int d^3r\, \phi\rho$$

If the charge density is due to the charged particles, then

$$\rho(\mathbf{r},t) = \sum_i e_i \delta(\mathbf{r} - \mathbf{r}_i), \quad \phi(\mathbf{r},t) = \int d^3r' \frac{\rho(\mathbf{r}',t)}{|\mathbf{r} - \mathbf{r}'|}$$

then

$$\int d^3r\, \mathbf{E}_L^2 = \frac{1}{\varepsilon_0}\int d^3r \int d^3r' \frac{\rho(\mathbf{r},t)\rho(\mathbf{r}',t)}{|\mathbf{r} - \mathbf{r}'|} = \frac{1}{\varepsilon_0}\sum_{i,j\neq i} \frac{e_i e_j}{|\mathbf{r}_i - \mathbf{r}_j|}$$

which is expressed entirely in terms of the particle variables.

The point of all of this: In the Coulomb gauge all quantities naturally separate into two distinct sets — one part describes the transverse field alone and the other can be expressed entirely in terms of the particle variables. In this gauge, when we want to quantize the field, we only need to quantize the transverse part while the longitudinal part can be quantized with the matter. That's why in quantum optics we always adopt the Coulomb gauge, which sometimes is also call the *radiation gauge*.

4.2 Free Space and Plane Wave Expansions

Now let us consider a classical EM field in empty space in the absence of any sources such as charges and currents. The vector potential then satisfies the homogeneous wave equation:

$$-\nabla^2 \mathbf{A} + \frac{1}{c^2}\frac{\partial^2 \mathbf{A}}{\partial t^2} = 0 \tag{4.16}$$

In order to obtain the Hamiltonian equations of motion, it is useful first to make a Fourier decomposition of $\mathbf{A}(\mathbf{r},t)$ with respect to its space variables x, y, z. Consider the EM field to be contained in a large cube of side L and we impose periodic boundary conditions and write

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{\sqrt{\varepsilon_0 V}}\sum_{\mathbf{k}} \boldsymbol{\mathcal{A}}_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{r}}$$

where $\mathbf{k} = (2\pi n_1/L, 2\pi n_2/L, 2\pi n_3/L)$ with $n_i = 0, \pm 1, \pm 2, ...$, and $V = L^3$ is the quantized volume. The transversality and reality of $\mathbf{A}$ require

$$\mathbf{k} \cdot \boldsymbol{\mathcal{A}}_{\mathbf{k}}(t) = 0, \quad \boldsymbol{\mathcal{A}}_{-\mathbf{k}}(t) = \boldsymbol{\mathcal{A}}^*_{\mathbf{k}}(t)$$

and the general solution of (4.16) is given by

$$\boldsymbol{\mathcal{A}}_{\mathbf{k}}(t) = \mathbf{c}_{\mathbf{k}} e^{-i\omega t} + \mathbf{c}^*_{-\mathbf{k}} e^{i\omega t}$$

with $\omega = ck$.

It is convenient to resolve the vector $\mathbf{c}_{\mathbf{k}}$ into two orthogonal components:

$$\mathbf{c}_{\mathbf{k}} = \sum_{s=1}^{2} c_{\mathbf{k}s} \boldsymbol{\epsilon}_{\mathbf{k}s}$$

where $\boldsymbol{\epsilon}_{\mathbf{k}s}$ $(s = 1, 2)$ are unit polarization vectors that obey the conditions

$$\mathbf{k} \cdot \boldsymbol{\epsilon}_{\mathbf{k}s} = 0, \quad \boldsymbol{\epsilon}^*_{\mathbf{k}s} \cdot \boldsymbol{\epsilon}_{\mathbf{k}s'} = \delta_{s,s'}, \quad \boldsymbol{\epsilon}_{\mathbf{k}1} \times \boldsymbol{\epsilon}_{\mathbf{k}2} = \mathbf{k}/k$$

which signify transversality, orthonormality and right-handedness, respectively.

With this definition, the vector potential can be expanded as

$$\begin{aligned} \mathbf{A}(\mathbf{r}, t) &= \frac{1}{\sqrt{\varepsilon_0 V}} \sum_{\mathbf{k},s} \left(c_{\mathbf{k}s} \boldsymbol{\epsilon}_{\mathbf{k}s}\, e^{-i\omega t} + c^*_{-\mathbf{k}s} \boldsymbol{\epsilon}^*_{-\mathbf{k}s}\, e^{i\omega t} \right) e^{i\mathbf{k}\cdot\mathbf{r}} \\ &= \frac{1}{\sqrt{\varepsilon_0 V}} \sum_{\mathbf{k},s} \left[u_{\mathbf{k}s}(t) \boldsymbol{\epsilon}_{\mathbf{k}s}\, e^{i\mathbf{k}\cdot\mathbf{r}} + u^*_{\mathbf{k}s}(t) \boldsymbol{\epsilon}^*_{\mathbf{k}s}\, e^{-i\mathbf{k}\cdot\mathbf{r}} \right] \end{aligned}$$

with $u_{\mathbf{k}s}(t) = c_{\mathbf{k}s}\, e^{-i\omega t}$. We can immediately write down the corresponding expansions for the EM fields:

$$\begin{aligned} \mathbf{E}(\mathbf{r}, t) &= -\frac{\partial}{\partial t} \mathbf{A}(\mathbf{r}, t) = \frac{i}{\sqrt{\varepsilon_0 V}} \sum_{\mathbf{k},s} \omega \left[u_{\mathbf{k}s}(t) \boldsymbol{\epsilon}_{\mathbf{k}s}\, e^{i\mathbf{k}\cdot\mathbf{r}} - c.c. \right] \\ \mathbf{B}(\mathbf{r}, t) &= \nabla \times \mathbf{A}(\mathbf{r}, t) = \frac{i}{\sqrt{\varepsilon_0 V}} \sum_{\mathbf{k},s} \left[u_{\mathbf{k}s}(t) (\mathbf{k} \times \boldsymbol{\epsilon}_{\mathbf{k}s})\, e^{i\mathbf{k}\cdot\mathbf{r}} - c.c. \right] \end{aligned}$$

These expressions allow us to write down the energy H of the EM field as

$$H = \frac{1}{2} \int d^3r \left[\varepsilon_0 \mathbf{E}^2(\mathbf{r}, t) + \frac{1}{\mu_0} \mathbf{B}^2(\mathbf{r}, t) \right] = 2 \sum_{\mathbf{k},s} \omega^2 |u_{\mathbf{k}s}(t)|^2 \tag{4.17}$$

which expresses the energy as a sum over the modes. In deriving 4.17), we have used

$$\frac{1}{V} \int d^3r\, e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}} = \delta_{\mathbf{k}\mathbf{k}'}, \quad (\mathbf{k} \times \boldsymbol{\epsilon}^*_{\mathbf{k}s}) \cdot (\mathbf{k} \times \boldsymbol{\epsilon}_{\mathbf{k}s'}) = k^2 \boldsymbol{\epsilon}^*_{\mathbf{k}s} \cdot \boldsymbol{\epsilon}_{\mathbf{k}s'} = k^2 \delta_{ss'}$$

4.3 Field Quantization

For the purpose of field quantization, it is desirable to write H in Hamiltonian form, which we do by introducing a pair of real canonical variables $q_{\mathbf{k}s}(t)$ and $p_{\mathbf{k}s}(t)$:

$$q_{\mathbf{k}s}(t) = u_{\mathbf{k}s}(t) + u^*_{\mathbf{k}s}(t), \quad p_{\mathbf{k}s}(t) = -i\omega \left[u_{\mathbf{k}s}(t) - u^*_{\mathbf{k}s}(t) \right]$$

in terms of which the energy becomes

$$H = \frac{1}{2}\sum_{\mathbf{k},s}\left[p^2_{\mathbf{k}s}(t) + \omega^2 q^2_{\mathbf{k}s}(t)\right] \tag{4.18}$$

This will be recognized as the energy of *a system of independent harmonic oscillators.* The canonical quations of motion are

$$\dot{q}_{\mathbf{k}s} = \partial H/\partial p_{\mathbf{k}s}, \quad -\dot{p}_{\mathbf{k}s} = \partial H/\partial q_{\mathbf{k}s}$$

In terms of the canonical variables, we have

$$\mathbf{A}(\mathbf{r},t) = \frac{1}{2\sqrt{\varepsilon_0 V}}\sum_{\mathbf{k},s}\left\{\left[q_{\mathbf{k}s}(t) + (i/\omega)p_{\mathbf{k}s}(t)\right]\boldsymbol{\epsilon}_{\mathbf{k}s}\, e^{i\mathbf{k}\cdot\mathbf{r}} + c.c.\right\} \tag{4.19}$$

$$\mathbf{E}(\mathbf{r},t) = \frac{i}{2\sqrt{\varepsilon_0 V}}\sum_{\mathbf{k},s}\left\{\left[\omega q_{\mathbf{k}s}(t) + ip_{\mathbf{k}s}(t)\right]\boldsymbol{\epsilon}_{\mathbf{k}s}\, e^{i\mathbf{k}\cdot\mathbf{r}} - c.c.\right\} \tag{4.20}$$

$$\mathbf{B}(\mathbf{r},t) = \frac{i}{2\sqrt{\varepsilon_0 V}}\sum_{\mathbf{k},s}\left\{\left[q_{\mathbf{k}s}(t) + (i/\omega)p_{\mathbf{k}s}(t)\right](\mathbf{k}\times\boldsymbol{\epsilon}_{\mathbf{k}s})\, e^{i\mathbf{k}\cdot\mathbf{r}} - c.c.\right\} \tag{4.21}$$

To quantize the field, we simply regard the canonical variables $q_{\mathbf{k}s}(t)$ and $p_{\mathbf{k}s}(t)$ as quantum operators $\hat{q}_{\mathbf{k}s}(t)$ and $\hat{p}_{\mathbf{k}s}(t)$ that obey the proper commutation relations:

$$[\hat{q}_{\mathbf{k}s}(t), \hat{p}_{\mathbf{k}'s'}(t)] = i\hbar\delta_{\mathbf{k}\mathbf{k}'}\delta_{ss'}, \quad [\hat{p}_{\mathbf{k}s}(t), \hat{p}_{\mathbf{k}'s'}(t)] = 0, \quad [\hat{q}_{\mathbf{k}s}(t), \hat{q}_{\mathbf{k}'s'}(t)] = 0$$

The energy in (4.18) should now be regarded as the Hamiltonian:

$$H = \frac{1}{2}\sum_{\mathbf{k},s}\left[\hat{p}^2_{\mathbf{k}s}(t) + \omega^2\hat{q}^2_{\mathbf{k}s}(t)\right]$$

The expansions in (4.19), (4.20) and (4.21) remain valid, but again, $\hat{\mathbf{A}}(\mathbf{r},t)$, $\hat{\mathbf{E}}(\mathbf{r},t)$ and $\hat{\mathbf{B}}(\mathbf{r},t)$ are now field operators.

For many purposes it is more convenient to deal, not with $\hat{q}_{\mathbf{k}s}(t)$ and $\hat{p}_{\mathbf{k}s}(t)$, but with a set of non-Hermitian operators defined by

$$\hat{a}_{\mathbf{k}s}(t) = \frac{1}{\sqrt{2\hbar\omega}}\left[\omega\hat{q}_{\mathbf{k}s}(t) + i\hat{p}_{\mathbf{k}s}(t)\right], \quad \hat{a}^\dagger_{\mathbf{k}s}(t) = \frac{1}{\sqrt{2\hbar\omega}}\left[\omega\hat{q}_{\mathbf{k}s}(t) - i\hat{p}_{\mathbf{k}s}(t)\right] \tag{4.22}$$

with the corresponding commutation relations

$$[\hat{a}_{\mathbf{k}s}(t), \hat{a}^\dagger_{\mathbf{k}'s'}(t)] = \delta_{\mathbf{k}\mathbf{k}'}\delta_{ss'}, \quad [\hat{a}_{\mathbf{k}s}(t), \hat{a}_{\mathbf{k}'s'}(t)] = 0, \quad [\hat{a}^\dagger_{\mathbf{k}s}(t), \hat{a}^\dagger_{\mathbf{k}'s'}(t)] = 0$$

In terms of $\hat{a}_{\mathbf{k}s}(t)$ and $\hat{a}^\dagger_{\mathbf{k}s}(t)$, we have

$$\hat{q}_{\mathbf{k}s}(t) = \sqrt{\hbar/(2\omega)}[\hat{a}_{\mathbf{k}s}(t) + \hat{a}^\dagger_{\mathbf{k}s}(t)], \quad \hat{p}_{\mathbf{k}s}(t) = -i\sqrt{\hbar\omega/2}[\hat{a}_{\mathbf{k}s}(t) - \hat{a}^\dagger_{\mathbf{k}s}(t)]$$

Hence apart from a factor $\sqrt{\hbar/(2\omega)}$, the operators $\hat{a}_{\mathbf{k}s}(t)$ and $\hat{a}^\dagger_{\mathbf{k}s}(t)$ correspond to the complex amplitude $u_{\mathbf{k}s}(t)$ and $u^*_{\mathbf{k}s}(t)$.

Finally, we can write the Hamiltonian in terms of $\hat{a}_{\mathbf{k}s}(t)$ and $\hat{a}^\dagger_{\mathbf{k}s}(t)$ as

$$H = \sum_{\mathbf{k},s}\hbar\omega\left[\hat{a}^\dagger_{\mathbf{k}s}(t)\hat{a}_{\mathbf{k}s}(t) + 1/2\right] \tag{4.23}$$

From (4.23) we can immediately obtain the equations of motion for operators $\hat{a}_{\mathbf{k}s}(t)$ and $\hat{a}^\dagger_{\mathbf{k}s}(t)$, and they can be easily solved as

$$\hat{a}_{\mathbf{k}s}(t) = \hat{a}_{\mathbf{k}s}(0)\, e^{-i\omega t}, \quad \hat{a}^\dagger_{\mathbf{k}s}(t) = \hat{a}^\dagger_{\mathbf{k}s}(0)\, e^{i\omega t}$$

In terms of these operators, the field operators can be written as

$$\hat{\mathbf{A}}(\mathbf{r},t) = \sum_{\mathbf{k},s} \sqrt{\frac{\hbar}{2\omega\varepsilon_0 V}} \left[\hat{a}_{\mathbf{k}s}(0)\boldsymbol{\epsilon}_{\mathbf{k}s}\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + h.c.\right] \tag{4.24}$$

$$\hat{\mathbf{E}}(\mathbf{r},t) = \sum_{\mathbf{k},s} \sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}} \left[i\hat{a}_{\mathbf{k}s}(0)\boldsymbol{\epsilon}_{\mathbf{k}s}\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + h.c.\right] \tag{4.25}$$

$$\hat{\mathbf{B}}(\mathbf{r},t) = \sum_{\mathbf{k},s} \sqrt{\frac{\hbar}{2\omega\varepsilon_0 V}} \left[i\hat{a}_{\mathbf{k}s}(0)(\mathbf{k}\times\boldsymbol{\epsilon}_{\mathbf{k}s})\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + h.c.\right] \tag{4.26}$$

The first term on the r.h.s. of the above equations is often denoted by $\hat{\mathbf{A}}^{(+)}(\mathbf{r},t)$, $\hat{\mathbf{E}}^{(+)}(\mathbf{r},t)$, $\hat{\mathbf{B}}^{(+)}(\mathbf{r},t)$, respectively, because it depends on positive frequencies. Correspondingly, the second terms are denoted by $\hat{\mathbf{A}}^{(-)}(\mathbf{r},t)$, $\hat{\mathbf{E}}^{(-)}(\mathbf{r},t)$, $\hat{\mathbf{B}}^{(-)}(\mathbf{r},t)$, respectively. The quantity $E_0 = \sqrt{\hbar\omega/(2\varepsilon_0 V)}$ is sometimes called the *electric field per photon.*

4.4 Physical Interpretation of $\hat{a}_{\mathbf{k}s}(t)$ and $\hat{a}^\dagger_{\mathbf{k}s}(t)$

The Hermitian operator $\hat{a}^\dagger_{\mathbf{k}s}\hat{a}_{\mathbf{k}s}$ that appears in the Hamiltonian (4.23) is a particularly important one and will be denoted by $\hat{n}_{\mathbf{k}s}$. It's easy to show that

$$[\hat{a}_{\mathbf{k}s}, \hat{n}_{\mathbf{k}'s'}] = \hat{a}_{\mathbf{k}s}\delta_{\mathbf{k}\mathbf{k}'}\delta_{ss'}, \quad [\hat{a}^\dagger_{\mathbf{k}s}, \hat{n}_{\mathbf{k}'s'}] = -\hat{a}^\dagger_{\mathbf{k}s}\delta_{\mathbf{k}\mathbf{k}'}\delta_{ss'}$$

Now let us examine the eigenvalues of $\hat{n}_{\mathbf{k}s}$, which will immediately lead us to those of H. If $n_{\mathbf{k}s}$ is an eigenvalue of $\hat{n}_{\mathbf{k}s}$, with corresponding eigenstate $|n_{\mathbf{k}s}\rangle$, i.e.

$$\hat{n}_{\mathbf{k}s}|n_{\mathbf{k}s}\rangle = n_{\mathbf{k}s}|n_{\mathbf{k}s}\rangle$$

then we have

$$\hat{n}_{\mathbf{k}s}\hat{a}^\dagger_{\mathbf{k}s}|n_{\mathbf{k}s}\rangle = \hat{a}^\dagger_{\mathbf{k}s}(\hat{n}_{\mathbf{k}s}+1)|n_{\mathbf{k}s}\rangle = (n_{\mathbf{k}s}+1)\hat{a}^\dagger_{\mathbf{k}s}|n_{\mathbf{k}s}\rangle$$

Hence the state $\hat{a}^\dagger_{\mathbf{k}s}|n_{\mathbf{k}s}\rangle$ is also an eigenstate of $\hat{n}_{\mathbf{k}s}$ with eigenvalue $(n_{\mathbf{k}s}+1)$, i.e.

$$\hat{a}^\dagger_{\mathbf{k}s}|n_{\mathbf{k}s}\rangle = g_{\mathbf{k}s}|n_{\mathbf{k}s}+1\rangle$$

where $g_{\mathbf{k}s}$ is a normalization constant. Take the norms of both sides of the above equation, we have

$$\langle n_{\mathbf{k}s}|\hat{a}_{\mathbf{k}s}\hat{a}^\dagger_{\mathbf{k}s}|n_{\mathbf{k}s}\rangle = |g_{\mathbf{k}s}|^2$$

On the other hand, $\hat{a}_{\mathbf{k}s}\hat{a}^\dagger_{\mathbf{k}s} = \hat{n}_{\mathbf{k}s}+1$, hence

$$\langle n_{\mathbf{k}s}|\hat{a}_{\mathbf{k}s}\hat{a}^\dagger_{\mathbf{k}s}|n_{\mathbf{k}s}\rangle = \langle n_{\mathbf{k}s}|(\hat{n}_{\mathbf{k}s}+1)|n_{\mathbf{k}s}\rangle = n_{\mathbf{k}s}+1$$

Therefore $|g_{\mathbf{k}s}|^2 = n_{\mathbf{k}s}+1$, and apart from a unimodular factor, we have

$$\hat{a}^\dagger_{\mathbf{k}s}|n_{\mathbf{k}s}\rangle = \sqrt{n_{\mathbf{k}s}+1}\,|n_{\mathbf{k}s}+1\rangle$$

This argument can be repeated and in general, we have

$$(\hat{a}^\dagger_{\mathbf{k}s})^r|n_{\mathbf{k}s}\rangle = \sqrt{(n_{\mathbf{k}s}+1)...(n_{\mathbf{k}s}+r+1)}\,|n_{\mathbf{k}s}+r\rangle, \quad r=1,2,3...$$

Hence the spectrum of eigenvalues of $\hat{n}_{\mathbf{k}s}$ is unbounded from above.

Apply a similar procedure to $\hat{a}_{\mathbf{k}s}|n_{\mathbf{k}s}\rangle$, we find

$$\hat{a}_{\mathbf{k}s}|n_{\mathbf{k}s}\rangle = \sqrt{n_{\mathbf{k}s}}\,|n_{\mathbf{k}s}\rangle$$

and in general

$$(\hat{a}_{\mathbf{k}s})^r|n_{\mathbf{k}s}\rangle = \sqrt{n_{\mathbf{k}s}(n_{\mathbf{k}s}-1)...(n_{\mathbf{k}s}-r+1)}\,|n_{\mathbf{k}s}-r\rangle, \quad r=1,2,3...$$

Now the sequence of numbers $n_{\mathbf{k}s}$, $n_{\mathbf{k}s}-1$, $n_{\mathbf{k}s}-2$ ..., must eventually become negative, yet the eigenvalues of $\hat{n}_{\mathbf{k}s}$ cannot in fact be negative (since $\langle\phi|\hat{n}_{\mathbf{k}s}|\phi\rangle = \langle\phi'|\phi'\rangle \geq 0$, with $|\phi'\rangle = \hat{a}_{\mathbf{k}s}|\phi\rangle$), the sequence of eigenvalues ..., $n_{\mathbf{k}s}-1$, $n_{\mathbf{k}s}$, $n_{\mathbf{k}s}+1$ ... must be bounded from below and cannot become negative. This requirement can only be reconciled if the lowest eigenvalue is zero such that

$$\hat{a}_{\mathbf{k}s}|0_{\mathbf{k}s}\rangle = 0$$

then the sequence terminates automatically. As a result the spectrum of $\hat{n}_{\mathbf{k}s}$ is therefore the set of non-negative integers 0, 1, 2, ...

For this reason, the operator $\hat{n}_{\mathbf{k}s}$ is known as the *number operator* for the ($\mathbf{k}s$) mode, and $\hat{a}_{\mathbf{k}s}(t)$ and $\hat{a}^\dagger_{\mathbf{k}s}(t)$ are respectively the corresponding *annihilation* and *creation* operators, and the excitations or quanta they annihilate/create are called *photons*.

Chapter 5

States of the Electromagnetic Field

5.1 Fock State

All the number operators $\hat{n}_{\mathbf{k}s}$ form a complete set of commuting observables for the field. Since the oprators corresponding to different modes $\mathbf{k}, s$ operate on different subspaces of Hilbert space, we can form a state vector characterizing the entire field by taking the drect product of $|n_{\mathbf{k}s}\rangle$ state vectors over all the modes: $|\{n\}\rangle = \prod_{\mathbf{k},s} |n_{\mathbf{k}s}\rangle$. Such a state is known as a *Fock state* of the electromagnetic field. The state $|\{0\}\rangle$ for which all the occupation numbers are zero is known as the *vacuum state* $|\text{vac}\rangle$. All the Fock states form a complete orthonormal basis, i.e., $1 = \sum_{\mathbf{k},s} |\{n\}\rangle\langle\{n\}|$.

For simplicity, from now on, we'll concentrate on a single mode situation, hence we'll drop the subscripts $\mathbf{k}, s$.

The Fock state $|n\rangle$ is an eigenstate of the number operator $\hat{n}$: $\hat{n}\,|n\rangle = n\,|n\rangle$. We can form any Fock state by repeated application of the creation operator $\hat{a}^\dagger$ on the vacuum:

$$|n\rangle = \frac{(\hat{a}^\dagger)^n}{\sqrt{n!}}\,|0\rangle$$

Using the expression for a single mode field

$$\hat{\mathbf{E}}(\mathbf{r},t) = E_0\left[i\hat{a}(0)\boldsymbol{\epsilon}\,e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)} + h.c.\right]$$

we immediately have

$$\langle n|\hat{\mathbf{E}}|n\rangle = 0, \quad \langle n|\hat{\mathbf{E}}^2|n\rangle = E_0^2(2n+1), \quad \langle n|(\Delta\hat{\mathbf{E}})^2|n\rangle = \langle n|\hat{\mathbf{E}}^2|n\rangle - \langle n|\hat{\mathbf{E}}|n\rangle^2 = E_0^2(2n+1)$$

Hence the electric field fluctuates even for a vacuum state with $n = 0$.

5.2 Coherent State

5.2.1 Definition

Classical EM field is characterized by a complex amplitude which contains information about the magnitude and the phase of the field. The analogous quantum state of the field is the *coherent state*, denoted by $|\alpha\rangle$,

which is an eigenstate of the annihilation operator with eigenvalue α:

$$\hat{a}\,|\alpha\rangle = \alpha\,|\alpha\rangle$$

Since $\hat{a}$ is not Hermitian, its eigenvalue will in general be complex.

Since the Fock states form a complete set, we have

$$|\alpha\rangle = \sum_{n=0}^{\infty} |n\rangle\langle n|\alpha\rangle = \sum_{n=0}^{\infty} |n\rangle\langle 0|\frac{\hat{a}^n}{\sqrt{n!}}|\alpha\rangle = \sum_{n=0}^{\infty} |n\rangle\langle 0|\frac{\alpha^n}{\sqrt{n!}}|\alpha\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle\langle 0|\alpha\rangle$$

Normalization condition requires

$$1 = \langle\alpha|\alpha\rangle = \sum_{n=0} \frac{|\alpha|^{2n}}{n!}\,|\langle 0|\alpha\rangle|^2 = e^{|\alpha|^2}|\langle 0|\alpha\rangle|^2$$

Hence apart from a unimodular factor, we have

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\,|n\rangle$$

The vacuum state $|0\rangle$ can either be regarded as a Fock state or a coherent state. The probability $p(n)$ that n photons will be found in the coherent state $|\alpha\rangle$ is given by

$$p(n) = |\langle n|\alpha\rangle|^2 = \frac{|\alpha|^{2n}}{n!}\,e^{-|\alpha|^2}$$

which will be recognized as a Poisson distribution in n. The mean number of photons is given by

$$\langle\alpha|\hat{n}|\alpha\rangle = |\alpha|^2$$

Also we have

$$\langle\alpha|\hat{n}^2|\alpha\rangle = |\alpha|^2 + |\alpha|^4 = \langle\hat{n}\rangle + \langle\hat{n}\rangle^2$$

Hence the number variance

$$\langle(\Delta\hat{n})\rangle = \langle\hat{n}^2\rangle - \langle\hat{n}\rangle^2 = \langle\hat{n}\rangle$$

For the coherent state, $p(n)$ peaks at $n = \langle\hat{n}\rangle$.

5.2.2 Displacement Operator

We can rewrite the Fock states expansion of a coherent state as

$$|\alpha\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}\,|n\rangle = e^{-|\alpha|^2/2} \sum_{n=0}^{\infty} \frac{\alpha^n(\hat{a}^\dagger)^n}{n!}\,|0\rangle = e^{-|\alpha|^2/2} e^{\alpha\hat{a}^\dagger}|0\rangle$$

Using the result $e^{-\alpha^*\hat{a}}|0\rangle = |0\rangle$, we can make the above expression more symmetric

$$|\alpha\rangle = e^{-|\alpha|^2/2} e^{\alpha\hat{a}^\dagger} e^{-\alpha^*\hat{a}}|0\rangle$$

We now make use of the Campbel-Baker-Hausdorff operator identity for two operators $\hat{A}$, $\hat{B}$

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}}\,e^{\hat{B}}\,e^{-[\hat{A},\hat{B}]/2}$$

provided that

$$[\hat{A},[\hat{A},\hat{B}]] = 0 = [\hat{B},[\hat{A},\hat{B}]]$$

Now put $\hat{A} = \alpha\hat{a}^\dagger$, $\hat{B} = -\alpha^*\hat{a}$ we have

$$e^{-|\alpha|^2/2}e^{\alpha\hat{a}^\dagger}e^{-\alpha^*\hat{a}} = e^{\alpha\hat{a}^\dagger - \alpha^*\hat{a}} \equiv \hat{D}(\alpha)$$

which allows us to have the following compact expression:

$$|\alpha\rangle = \hat{D}(\alpha)|0\rangle$$

$\hat{D}(\alpha)$ is the *displacement operator* that creates the coherent state $|\alpha\rangle$ from the vacuum state.

It is not difficult to derive the following properties of the displacement operator:

- $\hat{D}^\dagger(\alpha)\hat{D}(\alpha) = \hat{D}(\alpha)\hat{D}^\dagger(\alpha) = 1$
- $\hat{D}^\dagger(\alpha) = \hat{D}(-\alpha)$
- $\hat{D}^\dagger(\alpha)\hat{a}\hat{D}(\alpha) = \hat{a} + \alpha$, $\hat{D}^\dagger(\alpha)\hat{a}^\dagger\hat{D}(\alpha) = \hat{a}^\dagger + \alpha^*$
- $\hat{D}^\dagger(\alpha)f(\hat{a},\hat{a}^\dagger)\hat{D}(\alpha) = f(\hat{a}+\alpha,\hat{a}^\dagger+\alpha^*)$
- $\hat{D}(\alpha)\hat{D}(\beta) = e^{(\alpha\beta^* - \alpha^*\beta)/2}\hat{D}(\alpha+\beta)$ Note that $\alpha\beta^* - \alpha^*\beta$ is purely imaginary, hence the factor in front of $\hat{D}(\alpha+\beta)$ is simply a phase factor.
- Two different displacement operators $\hat{D}(\alpha)$ and $\hat{D}(\beta)$ are orthogonal in the sense that

  $$\mathrm{Tr}[\hat{D}(\alpha)\hat{D}^\dagger(\beta)] = \pi\delta^2(\alpha-\beta)$$

  where $\delta^2(v) = \delta(\mathrm{Re}[v])\delta(\mathrm{Im}[v])$.

5.2.3 Time Evolution and Uncertainty Products

In the Schrödinger picture, we have

$$|\psi(t)\rangle = e^{-iHt/\hbar}|\psi(0)\rangle$$

Our Hamiltonian is $H = \hbar\omega(\hat{n}+1/2)$. If we start in a coherent state $|\psi(0)\rangle = |\alpha\rangle$, then we have

$$|\psi(t)\rangle = e^{-i\omega t/2}\,e^{-i\omega t\hat{n}}|\alpha\rangle = e^{-i\omega t/2}\,e^{-|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}e^{-i\omega t\hat{n}}|n\rangle = e^{-i\omega t/2}|\alpha e^{-i\omega t}\rangle$$

Apart from the phase factor, this is just another coherent state. Following this, we immediately have

$$\langle\hat{a}\rangle = \alpha e^{-i\omega t}, \quad \langle\hat{a}^\dagger\rangle = \alpha^* e^{i\omega t}$$

where the expectation value is w.r.t. $|\psi(t)\rangle$. Going back to the canonically conjugate operators $\hat{p}$ and $\hat{q}$, we have

$$\langle\hat{q}\rangle = \sqrt{\frac{\hbar}{2\omega}}\left(\alpha e^{-i\omega t} + c.c.\right), \quad \langle\hat{p}\rangle = -i\sqrt{\frac{\hbar\omega}{2}}\left(\alpha e^{-i\omega t} - c.c.\right)$$

These are reminiscent of the motion of a classical harmonic oscillator of frequency ω, having a well-defined complex amplitude α.

It is also easy to show that

$$\langle \hat{q}^2 \rangle = \frac{\hbar}{2\omega} \left[\alpha^2 e^{-i2\omega t} + (\alpha^*)^2 e^{i2\omega t} + 2\alpha^* \alpha + 1 \right]$$

Hence the variance

$$\langle (\Delta \hat{q})^2 \rangle = \langle \hat{q}^2 \rangle - \langle \hat{q} \rangle^2 = \frac{\hbar}{2\omega}$$

Similarly, we can also find that

$$\langle (\Delta \hat{p})^2 \rangle = \frac{\hbar\omega}{2}$$

Thus

$$\sqrt{\langle (\Delta \hat{q})^2 \rangle \langle (\Delta \hat{p})^2 \rangle} = \hbar/2$$

which is the minimum value allowed by the Heisenberg uncertainty relationship. Therefore in a coherent state, the canonical variables $\hat{p}$ and $\hat{q}$ are as well defined as quantum mechanics allows.

5.2.4 Coherent States as a Basis

Coherent states form a basis for the representation of arbitrary quantum states. Because the coherent states are eigenstates of a non-Hermitian operator, the coherent-state representation has some unusual features.

First, no two coherent states are ever orthogonal. This can be easily show as

$$\langle \alpha | \beta \rangle = e^{-(|\alpha|^2 + |\beta|^2)/2} \sum_n \frac{(\alpha^* \beta)^n}{n!} = e^{-(|\alpha|^2 + |\beta|^2 - 2\alpha^* \beta)/2} = e^{-|\alpha - \beta|^2/2} \, e^{(\alpha^* \beta - \alpha \beta^*)/2}$$

The last term is a unimodular phase factor, thus

$$|\langle \alpha | \beta \rangle|^2 = e^{-|\alpha - \beta|^2}$$

Despite their non-orthogonality, the coherent states span the whole Hilbert space of state vectors as one can show that the identity operator 1 can be written as

$$1 = \frac{1}{\pi} \int |\alpha\rangle\langle\alpha| d^2\alpha$$

To show this, let us write $\alpha = re^{i\theta}$, so that $d^2\alpha = r dr d\theta$, and making use of the Fock state expansion of the coherent state, we have

$$\frac{1}{\pi} \int |\alpha\rangle\langle\alpha| d^2\alpha = \frac{1}{\pi} \int_0^\infty dr \int_0^{2\pi} d\theta \sum_{m,n} e^{-r^2} \frac{r^{n+m+1}}{\sqrt{n!\, m!}} \, e^{i(n-m)\theta} \, |n\rangle\langle m|$$

The integration over θ gives $2\pi\delta_{mn}$, therefore we have

$$\frac{1}{\pi} \int |\alpha\rangle\langle\alpha| d^2\alpha = \sum_n \frac{1}{n!} |n\rangle\langle n| \int_0^\infty dr \, 2r^{2n+1} e^{-r^2} = \sum_n |n\rangle\langle n| = 1$$

The set of coherent states is said to be *over-complete*, in the sense that the states form a basis and yet are expressible in terms of each other. Any arbitrary state $|\psi\rangle$ and arbitrary operator $\hat{A}$ can be expanded as

$$|\psi\rangle = \frac{1}{\pi} \int |\alpha\rangle\langle\alpha|\psi\rangle d^2\alpha, \quad \hat{A} = \frac{1}{\pi^2} \int \int \langle\alpha|\hat{A}|\beta\rangle |\alpha\rangle\langle\beta| \, d^2\alpha \, d^2\beta$$

5.2.5 Generation of Coherent States

Coherent states can be generated by classical c-number currents in the same way as a classical c-number force drives the single mode harmonic oscillator. Consider a quantum electromagnetic field of vector potential $\hat{\mathbf{A}}(\mathbf{r},t)$ that is interacting with a classical electric current with vector current density $\mathbf{j}(bfr,t)$. The interaction Hamiltonian is given by

$$H_I(t) = -\int d^3r\, \mathbf{j}(\mathbf{r},t)\cdot\hat{\mathbf{A}}(\mathbf{r},t) = -\sum_{\mathbf{k},s}\sqrt{\frac{\hbar}{2\omega\varepsilon_0 V}}\left[\boldsymbol{\epsilon}_{\mathbf{k}s}\cdot\mathbf{J}(\mathbf{k},t)\hat{a}_{\mathbf{k}s}\,e^{-i\omega t}+h.c.\right]$$

where $\mathbf{J}(\mathbf{k},t)=\int d^3r\,\mathbf{j}(\mathbf{r},t)e^{i\mathbf{k}\cdot\mathbf{r}}$ is the Fourier transform of the current density. The time-evolution operator in the interaction picture is then given by

$$U(t,0)=\exp\left[-i\int_0^t d\tau\, H_I(\tau)/\hbar\right]=\Pi_{\mathbf{k},s}\hat{D}\left[\alpha_{\mathbf{k}s}(t)\right]$$

with

$$\alpha_{\mathbf{k}s}(t)=-\frac{i}{\hbar}\sqrt{\frac{\hbar}{2\omega\varepsilon_0 V}}\int_0^t d\tau\,\boldsymbol{\epsilon}_{\mathbf{k}s}\cdot\mathbf{J}(\mathbf{k},\tau)e^{-i\omega\tau}$$

If the initial state is a vacuum state, we have

$$|\psi(t)\rangle = U(t,0)|\{0\}\rangle=\Pi_{\mathbf{k},s}\hat{D}\left[\alpha_{\mathbf{k}s}(t)\right]|\{0\}\rangle=|\{\alpha_{\mathbf{k}s}(t)\}\rangle$$

Thus, as promised, the c-number current generates a coherent state with the same eigenvalue as the classical field.

5.3 Squeezed State

Squeezed state attracted tremendous attention in the 1980's and early 1990's due to the promise of reduced noise in communication and improvements in precision measurements. Unfortunately these promises were not realized, but maybe they will in the future. The idea of squeezing was originally proposed relating to reducing quantum fluctuations in one of the field quadratures. But now it has been generalized to other arena. One very important concept in atomic physics is spin squeezing. Here we'll just focus on the quadrature squeezing of light field.

5.3.1 Definition of Quadrature Squeezing

Define two dimensionless variables $\hat{Q}'$ and $\hat{P}'$ as

$$\hat{Q}'=\hat{a}^\dagger+\hat{a},\quad \hat{P}'=i(\hat{a}^\dagger-\hat{a})$$

Readily we find $[\hat{Q}',\hat{P}']=2i$, so that $\hat{Q}'$ and $\hat{P}'$ behave like dimensionless canonical conjugates, and $\sqrt{\langle(\Delta\hat{Q}')^2\rangle\langle(\Delta\hat{P}')^2\rangle}\geq 1$. Another interpretation of $\hat{Q}'$ and $\hat{P}'$ is suggested if we express the electric field for a single-mode, linearly polarized light in the form

$$\hat{\mathbf{E}}(\mathbf{r},t)=l(\omega)\boldsymbol{\epsilon}[\hat{a}\,e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}+\hat{a}^\dagger\,e^{-i(\mathbf{k}\cdot\mathbf{r}-\omega t)}]=l(\omega)\boldsymbol{\epsilon}[\hat{Q}'\cos(\mathbf{k}\cdot\mathbf{r}-\omega t)-\hat{P}'\sin(\mathbf{k}\cdot\mathbf{r}-\omega t)]$$

Hence $\hat{Q}'$ and $\hat{P}'$ are also the amplitudes of the quadratures into which the oscillating field can be decomposed.

It is easy to find that for coherent states, we find that the variance of dispersion of the quadrature amplitudes are equal, with

$$\langle(\Delta\hat{Q}')^2\rangle = \langle(\Delta\hat{P}')^2\rangle = 1$$

For this state we can find a phase space distribution of $\hat{Q}'$ and $\hat{P}'$ which has circular symmetry, and the uncertainty product has its minimum value. If there exists a state for which either $\hat{Q}'$ or $\hat{P}'$ has a dispersion below unity, i.e., below the vacuum level, at the cost of a corresponding increase in the dispersion of the other variable, then we call the corresponding state a *squeezed state*. Squeezing need not be confined to the $\hat{Q}'$ and $\hat{P}'$ variables, rather it can extended to more general variables $\hat{Q}$ and $\hat{P}$ defined as

$$\hat{Q} = \hat{a}^\dagger\, e^{i\beta} + \hat{a}\, e^{-i\beta} = \hat{X}_\beta, \quad \hat{P} = \hat{a}^\dagger\, e^{i(\beta+\pi/2)} + \hat{a}\, e^{-i(\beta+\pi/2)} = \hat{X}_{\beta+\pi/2}$$

For coherent state, we have

$$\langle(\Delta\hat{X}_\beta^2\rangle = \langle(\Delta\hat{X}_{\beta+\pi/2})^2\rangle = 1$$

for *any* phase angle β. A squeezed state is defined more generally by the condition that there exists an angle θ for which the dispersion $\langle(\Delta\hat{Q})^2\rangle$ is smaller than in the vacuum state, i.e., $\langle(\Delta\hat{Q})^2\rangle < 1$. Such squeezing phenomenon is also sometimes referred to as *quadrature squeezing*, as there exist other types of squeezing.

5.3.2 Squeeze Operator

A squeezed state can be generated from an unsqueezed one by the action of the following unitary operator

$$\hat{S}(\xi) = e^{(\xi^*\hat{a}^2 - \xi\hat{a}^{\dagger 2})/2}, \;\; \xi = re^{i\theta}$$

which is known as the *squeeze operator*. It's easy to show that $\hat{S}^\dagger(\xi) = \hat{S}^{-1}(\xi) = \hat{S}(-\xi)$ and

$$\hat{S}^\dagger(\xi)\hat{a}\hat{S}(\xi) = \hat{a}\,\cosh r - \hat{a}^\dagger\, e^{i\theta}\sinh r, \quad \hat{S}^\dagger(\xi)\hat{a}^\dagger\hat{S}(\xi) = \hat{a}^\dagger\,\cosh r - \hat{a}\, e^{-i\theta}\sinh r \tag{5.1}$$

Such transformation is called the *Bogoliubov transformation*. The first of these can be shown as follows: Let $\hat{A} = -(\xi^*\hat{a}^2 - \xi\hat{a}^{\dagger 2})/2$, then $S(\xi) = e^{-\hat{A}}$ and $\hat{S}^\dagger(\xi)\hat{a}\hat{S}(\xi) = e^{\hat{A}}\hat{a}e^{-\hat{A}}$. According to the Operator Expansion Theorem,

$$e^{\hat{A}}\hat{a}e^{-\hat{A}} = \hat{a} + [\hat{A},\hat{a}] + \frac{1}{2!}[\hat{A},[\hat{A},\hat{a}]] + ...$$

Using

$$[\hat{A},\hat{a}] = -\xi\hat{a}^\dagger, \quad [\hat{A},\hat{a}^\dagger] = -\xi^*\hat{a}$$

we have

$$\hat{S}^\dagger(\xi)\hat{a}\hat{S}(\xi) = \left(\sum_{\text{even } n}\frac{r^n}{n!}\right)\hat{a} - \frac{\xi}{r}\left(\sum_{\text{odd } n}\frac{r^n}{n!}\right)\hat{a}^\dagger = \hat{a}\,\cosh r - \hat{a}^\dagger\, e^{i\theta}\sinh r$$

The *squeezed coherent state* is defined as

$$|\xi,\alpha\rangle \equiv \hat{S}(\xi)|\alpha\rangle = \hat{S}(\xi)\hat{D}(\alpha)|0\rangle$$

In particular, the state $|\xi,0\rangle = \hat{S}(\xi)|0\rangle$ is called the *squeezed vacuum*.

Using (5.1), we have

$$\begin{aligned}\hat{S}(\xi)\hat{X}_\beta\hat{S}(\xi) &= \hat{S}^\dagger(\xi)\hat{a}^\dagger\hat{S}(\xi)e^{i\beta} + \hat{S}^\dagger(\xi)\hat{a}\hat{S}(\xi)e^{-i\beta} \\ &= \left(\hat{a}^\dagger\,\cosh r - \hat{a}\,e^{-i\theta}\sinh r\right)e^{i\beta} + \left(\hat{a}\,\cosh r - \hat{a}^\dagger\,e^{i\theta}\sinh r\right)e^{-i\beta}\end{aligned}$$

In particular, when $\beta = \theta/2$, we have

$$\hat{S}^\dagger(\xi)\hat{X}_{\theta/2}\hat{S}(\xi) = e^{-r}\hat{X}_{\theta/2}$$

and

$$\hat{S}^\dagger(\xi)\hat{X}^2_{\theta/2}\hat{S}(\xi) = \hat{S}^\dagger(\xi)\hat{X}_{\theta/2}\hat{S}(\xi)\,\hat{S}(\xi)\hat{X}_{\theta/2}\hat{S}(\xi) = e^{-2r}\hat{X}^2_{\theta/2}$$

Then, w.r.t. the squeezed coherent state $|\xi,\alpha\rangle$, we have

$$\begin{aligned}\langle\xi,\alpha|(\Delta\hat{X}_{\theta/2})^2|\xi,\alpha\rangle &= \langle\xi,\alpha|\hat{X}^2_{\theta/2}|\xi,\alpha\rangle - \langle\xi,\alpha|\hat{X}_{\theta/2}|\xi,\alpha\rangle^2 \\ &= \langle\alpha|\hat{S}^\dagger(\xi)\hat{X}^2_{\theta/2}\hat{S}(\xi)|\alpha\rangle - \langle\alpha|\hat{S}^\dagger(\xi)\hat{X}_{\theta/2}\hat{S}(\xi)|\alpha\rangle^2 \\ &= e^{-2r}\,\langle\alpha|(\Delta\hat{X}_{\theta/2})^2|\alpha\rangle \\ &= e^{-2r} < 1\end{aligned}$$

And similarly, we have

$$\langle\xi,\alpha|(\Delta\hat{X}_{\theta/2+\pi/2})^2|\xi,\alpha\rangle = e^{2r} > 1$$

Thus the dispersion in $\hat{X}_{\theta/2}$ is below its vacuum value, that in $\hat{X}_{\theta/2+\pi/2}$ is above the vacuum value, but the product of the two remains unity and has the minimum value. The parameter r is called the *squeezing parameter*.

5.3.3 Two-mode Squeezed State

We can generalize the concept of the single-mode squeezed state to a two-mode one. The two-mode squeezed operator is defined as

$$\hat{S}_{AB}(\xi) = e^{\xi^*\hat{a}\hat{b} - \xi\hat{a}^\dagger\hat{b}^\dagger},\;\; \xi = re^{i\theta}$$

where $\hat{a}$ and $\hat{b}$ are annihilation operators for mode A and B, respectively. The two modes are independent from each other. The operator $\hat{S}_{AB}(\xi)$ is unitary since $\hat{S}^\dagger_{AB}(\xi) = \hat{S}^{-1}_{AB}(\xi) = \hat{S}_{AB}(-\xi)$, and we can show that

$$\hat{S}^\dagger_{AB}(\xi)\hat{a}\hat{S}_{AB}(\xi) = \hat{a}\cosh r - \hat{b}^\dagger e^{i\theta}\sinh r,\quad \hat{S}^\dagger_{AB}(\xi)\hat{b}\hat{S}_{AB}(\xi) = \hat{b}\cosh r - \hat{a}^\dagger e^{i\theta}\sinh r$$

We can express the two-mode squeezed state as a product of two single-mode squeezed states b introducing new modes C and D, whose annihilation operators are defined as

$$\hat{c} = \frac{1}{\sqrt{2}}\left[\hat{a} + e^{i\delta}\hat{b}\right],\quad \hat{d} = \frac{1}{\sqrt{2}}\left[\hat{b} - e^{i\delta}\hat{a}\right]$$

It's easy to show that $[\hat{c},\hat{c}^\dagger] = 1 = [\hat{c},\hat{c}^\dagger]$ and the modes C and D are independent from each other since $\hat{c}$ and $\hat{c}^\dagger$ commute with $\hat{d}$ and $\hat{d}^\dagger$.

In terms of $\hat{c}$ and $\hat{d}$, we have

$$\hat{S}_{AB}(\xi) = \hat{S}_C(\xi e^{i\delta})\,\hat{S}_D(-\xi e^{-i\delta})$$

One can show that for state $|\xi_{AB}\rangle = \hat{S}_{AB}(\xi)|0\rangle$, we have

$$\langle \hat{n}_A \rangle = \langle \hat{n}_B \rangle = \sinh^2 r = \bar{n}, \quad \langle (\Delta \hat{n}_A)^2 \rangle = \langle (\Delta \hat{n}_B)^2 \rangle = \bar{n}(\bar{n}+1)$$

but the variance in the difference between the photon numbers vanishes, i.e.,

$$\langle (\Delta(\hat{n}_A - \hat{n}_B))^2 \rangle = \langle (\Delta \hat{n}_A)^2 \rangle + \langle (\Delta \hat{n}_B)^2 \rangle - 2\left(\langle \hat{n}_A \hat{n}_B \rangle - \langle \hat{n}_A \rangle \langle \hat{n}_B \rangle\right) = 0$$

5.4 Thermal State

In a thermal state, we only know the mean energy of the system $E = \mathrm{Tr}[\rho \hat{H}]$, and that the entropy of the system takes maximum value. Given the density operator ρ, the von Neumann entropy is defined as

$$S = -k_B \mathrm{Tr}[\rho \ln \rho]$$

Hence in order to find the expression for ρ, we need to maximize S with the constraints $\mathrm{Tr}[\rho \hat{H}] = E$ and $\mathrm{Tr}[\rho] = 1$. Using the Langrange variational method with constraints, we have

$$\delta S = -k_B \mathrm{Tr}[(1 + \ln \rho + \beta \hat{H} + \lambda)(\delta \rho)] = 0$$

which yields

$$1 + \ln \rho + \beta \hat{H} + \lambda = 0, \quad \text{or} \quad \rho = e^{-(1+\lambda)}\, e^{-\beta \hat{H}}$$

Requirement of $\mathrm{Tr}[\rho] = 1$ yields

$$e^{1+\lambda} = \mathrm{Tr}[e^{-\beta \hat{H}}] \equiv Z$$

Z is known as the *partition function* and β can be identified as $k_B T$. Hence we have for any state in thermal equilibrium

$$\rho = \frac{1}{Z}\, e^{-\beta \hat{H}} \quad \text{with} \quad \beta = k_B T$$

Now consider a single-mode quantized optical field, $\hat{H} = \hbar\omega(\hat{n} + 1/2)$. We then have

$$\rho = \frac{e^{-\beta \hbar \omega \hat{n}}}{\mathrm{Tr}[e^{-\beta \hbar \omega \hat{n}}]} = \left(1 - e^{-\beta \hbar \omega}\right)\, e^{-\beta \hbar \omega \hat{n}}$$

where we have used

$$\mathrm{Tr}[f(\hat{n})] = \sum_n \langle n | f(\hat{n}) | n \rangle = \sum_n f(n)$$

and $\sum_n e^{-\beta \hbar \omega n} = (1 - e^{-\beta \hbar \omega})^{-1}$.

Using the Fock state basis, we can express the density operator as

$$\rho = \sum_n \left(1 - e^{-\beta \hbar \omega}\right)\, e^{-\beta \hbar \omega n} |n\rangle\langle n| = \sum_n p(n) |n\rangle\langle n|$$

where

$$p(n) = \left(1 - e^{-\beta \hbar \omega}\right)\, e^{-\beta \hbar \omega n} \tag{5.2}$$

is the probability for n photons in the mode. Eq. (12.2) represents the Bose-Einstein distribution.

It is straightforward to show that the mean photon number and the mean energy are

$$\langle \hat{n} \rangle = \frac{1}{e^{\beta\hbar\omega} - 1}, \quad \langle \hat{H} \rangle = \frac{\hbar\omega}{2} + \frac{\hbar\omega}{e^{\beta\hbar\omega} - 1}$$

Then at $T = 0$, $\langle \hat{H} \rangle = \hbar\omega/2$; while for $k_B T \gg \hbar\omega$, $\langle \hat{H} \rangle = k_B T$. We can express the probability $p(n)$ in (12.2) directly in terms of $\langle \hat{n} \rangle$ as

$$p(n) = \frac{\langle \hat{n} \rangle^n}{(1 + \langle \hat{n} \rangle)^{n+1}} = \frac{1}{1 + \langle \hat{n} \rangle} \left(\frac{\langle \hat{n} \rangle}{1 + \langle \hat{n} \rangle} \right)^n$$

Obviously, $p(n)$ is a monotonically decreasing function of n.

We can also derive the spectral distribution function. The number of states in volume V and in differential interval d^3k is $2\frac{V}{(2\pi)^3} d^3k$ where the factor of 2 arises from the two possible polarizations for each $\mathbf{k}$. After putting $d^3k = 4\pi(\omega^2/c^3)d\omega$, the density of photons with frequency range ω and $\omega + d\omega$ is

$$\Phi(\omega)d\omega = \langle \hat{n} \rangle 2 \frac{1}{(2\pi)^3} \frac{4\pi\omega^2}{c^3} d\omega = \frac{\omega^2}{\pi^2 c^3 (e^{\beta\hbar\omega} - 1)} d\omega$$

This is just the Planck's law for blackbody radiation.

Chapter 6

Beam Splitter and Homodyne Detection

The temporal fluctuation properties of light beams are measured by optical interference experiments. For example, the Mach-Zehnder interferometer and the Brown-Twiss interferometers can be used to measure the first- and second-order correlation functions, respectively. The central components in these experiments are optical beam splitters. Beam splitters also play important roles in studies of quantum aspects of light. For simplicity, in our discussion, we will assume a lossless beam splitter.

6.1 Lossless Beam Splitter

6.1.1 Classical Treatment

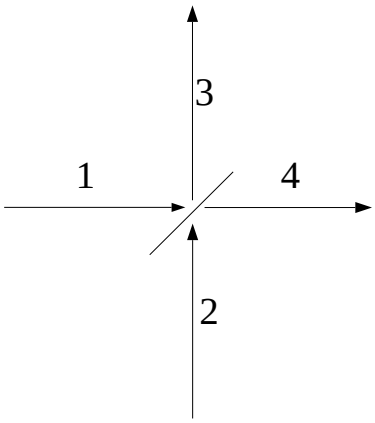

Figure 6-1: Light detection experiment.

Consider a beam splitter with two input fields E_1 and E_2, and two output fields E_3 and E_4. The output

fields are related to the input ones as

$$\begin{pmatrix} E_3 \\ E_4 \end{pmatrix} = \begin{pmatrix} \mathcal{R}_{31} & \mathcal{T}_{32} \\ \mathcal{T}_{41} & \mathcal{R}_{42} \end{pmatrix} \begin{pmatrix} E_1 \\ E_2 \end{pmatrix} \tag{6.1}$$

The 2×2 matrix is known as the *beam-splitter matrix*.

Energy conservation requires that $|E_1|^2 + |E_2|^2 = |E_3|^2 + |E_4|^2$ which yields

$$|\mathcal{R}_{31}|^2 + |\mathcal{T}_{41}|^2 = |\mathcal{R}_{42}|^2 + |\mathcal{T}_{32}|^2 = 1 \quad \text{and} \quad \mathcal{R}_{31}\mathcal{T}_{32}^* + \mathcal{T}_{41}\mathcal{R}_{42}^* = 0$$

If we write

$$\mathcal{R}_{31} = |\mathcal{R}_{31}|e^{i\phi_{31}}$$

and similarly for other 3 quantities, we can reexpress the above condition as

$$|\mathcal{R}_{31}| = |\mathcal{R}_{42}| = |\mathcal{R}|, \quad |\mathcal{T}_{32}| + |\mathcal{T}_{41}| = |\mathcal{T}| = \sqrt{1 - |\mathcal{R}|^2}, \quad \phi_{31} + \phi_{42} - \phi_{32} - \phi_{41} = \pm\pi$$

The lossless beam-splitter matrix is then unitary. We can usually choose the following symmetric coefficients

$$\mathcal{R}_{31} = \mathcal{R}_{42} = \mathcal{R} = |\mathcal{R}|e^{i\phi_{\mathcal{R}}}, \quad \mathcal{T}_{32} = \mathcal{T}_{41} = \mathcal{T} = |\mathcal{T}|e^{i\phi_{\mathcal{T}}}$$

with $|\mathcal{R}|^2 + |\mathcal{T}|^2 = 1$ and $\phi_{\mathcal{R}} - \phi_{\mathcal{T}} = \pm\pi/2$. For a 50:50 beam splitter, $|\mathcal{R}| = |\mathcal{T}| = 1/\sqrt{2}$.

6.1.2 Quantum Treatment

Quantum mechanically, the beam-splitter matrix is still valid, only that we have to replace the complex amplitude by operators:

$$\begin{pmatrix} \hat{a}_3 \\ \hat{a}_4 \end{pmatrix} = \begin{pmatrix} \mathcal{R} & \mathcal{T} \\ \mathcal{T} & \mathcal{R} \end{pmatrix} \begin{pmatrix} \hat{a}_1 \\ \hat{a}_2 \end{pmatrix} \tag{6.2}$$

If the two input fields are independent, i.e.,

$$[\hat{a}_1, \hat{a}_1^\dagger] = 1 = [\hat{a}_2, \hat{a}_2^\dagger], \quad [\hat{a}_1, \hat{a}_2^\dagger] = 0 = [\hat{a}_2, \hat{a}_1^\dagger]$$

then it is easy to see that

$$[\hat{a}_3, \hat{a}_3^\dagger] = 1 = [\hat{a}_4, \hat{a}_4^\dagger], \quad [\hat{a}_3, \hat{a}_4^\dagger] = 0 = [\hat{a}_4, \hat{a}_3^\dagger]$$

Define photon number operators as $\hat{n}_i = \hat{a}_i^\dagger \hat{a}_i$, using

$$\begin{aligned} \hat{n}_3 &= |\mathcal{R}|^2\hat{n}_1 + |\mathcal{T}|^2\hat{n}_2 + \mathcal{R}^*\mathcal{T}\hat{a}_1^\dagger\hat{a}_2 + \mathcal{R}\mathcal{T}^*\hat{a}_1\hat{a}_2^\dagger \\ \hat{n}_4 &= |\mathcal{T}|^2\hat{n}_1 + |\mathcal{R}|^2\hat{n}_2 + \mathcal{R}\mathcal{T}^*\hat{a}_1^\dagger\hat{a}_2 + \mathcal{R}^*\mathcal{T}\hat{a}_1\hat{a}_2^\dagger \end{aligned}$$

we can readily verify:

$$\hat{n}_1 + \hat{n}_2 = \hat{n}_3 + \hat{n}_4$$

which expresses the conservation of photons between the inputs and outputs.

We can already see the difference between the classical and quantum cases. If one of the input fields, say Field 2, is in the vacuum state, in the classical treatment, we can let $E_2 = 0$; quantum mechanically, however, we can never neglect $\hat{a}_2$ as we will show here.

If the input field 2 is in the vacuum state, it's easy to show that photon number fluctuations in the output fields are

$$\langle(\Delta\hat{n}_3)^2\rangle = |\mathcal{R}|^4\langle(\Delta\hat{n}_1)^2\rangle + |\mathcal{R}|^2|\mathcal{T}|^2\langle\hat{n}_1\rangle, \quad \langle(\Delta\hat{n}_4)^2\rangle = |\mathcal{T}|^4\langle(\Delta\hat{n}_1)^2\rangle + |\mathcal{R}|^2|\mathcal{T}|^2\langle\hat{n}_1\rangle$$

The second terms at the r.h.s. arises from the effect of the vacuum fluctuation injected at port 2. We can also show that the joint probability of detecting a photon in both the reflected and the transmitted beams is proportional to

$$P_{34} = \langle\hat{a}_3^\dagger\hat{a}_4^\dagger\hat{a}_4\hat{a}_3\rangle = \langle\hat{n}_3\hat{n}_4\rangle = |\mathcal{R}|^2|\mathcal{T}|^2\left(\langle\hat{n}_1^2\rangle - \langle\hat{n}_1\rangle\right)$$

If the input state at port 1 is the one-photon Fock state, then $\langle\hat{n}_1^2\rangle = \langle\hat{n}_1\rangle = 1$, then $P_{34} = 0$. In fact P_{34} vanishes when the input state at port 1 is any linear superposition of the vacuum state and the one photon Fock state $\alpha|0\rangle + \beta|1\rangle$. The probability of detecting a photon both in the reflected and in the transmitted beam is then zero, a consequence of the non-divisibility of a single photon. This conclusion is of course without analogy for a classical field.

If both of the input fields are in the one-photon Fock state, then one can show that

$$P_{34} = \left(|\mathcal{T}|^2 - |\mathcal{R}|^2\right)^2$$

This vanishes for a 50:50 beam splitter with $|\mathcal{T}| = |\mathcal{R}| = 1/\sqrt{2}$. Therefore when two photons enter a 50:50 beam splitter, one at each input port, we will never encounter one photon exiting at each output port; either both photons exit at port 3 or both exit at port 4. This is an example of quantum interference of the probability amplitudes for a photon pair and can be understood as follows. There are two different ways in which the situation with one photon exiting port 3 and the other exiting port 4 can arise. Either the two photons are both reflected or both transmitted. These two ways cannot be distinguished and hence their probability amplitude must be added. Due to the phase shifts associated with reflection and transmission, the two amplitudes are exactly π out of phase with each other and therefore cancel with each other. This effect can be employed to determine the time separation between two photons.

6.2 Homodyne Detection

Ordinary photodetectors detect light intensity or photon flux, homodyne detection by contrast measures the expectation values of the electric field quadrature operators. It is a particularly important technique for the study of phase sensitive phenomena.

In homodyne detection, the input field 1 of the beam splitter is the weak signal field and the input field 2 is taken to be strong coherent light field called *local oscillator* with state vector $||\alpha_L|e^{i\phi_L}\rangle$. Taking the phase convention such that $\mathcal{R} = iR$ and $\mathcal{T} = T$ where R and T are positive real numbers. Then we have

$$\begin{aligned}
\hat{n}_3 &= R^2\hat{n}_1 + T^2\hat{n}_2 + iRT(\hat{a}_2^\dagger\hat{a}_1 - \hat{a}_1^\dagger\hat{a}_2) \\
\hat{n}_4 &= T^2\hat{n}_1 + R^2\hat{n}_2 - iRT(\hat{a}_2^\dagger\hat{a}_1 - \hat{a}_1^\dagger\hat{a}_2)
\end{aligned}$$

6.2.1 Ordinary Homodyne Detection

In ordinary homodyne detection, we have $T^2 \gg R^2$, and the measured signal is the photon flux at output field 4, i.e., $\langle \hat{n}_4 \rangle$ which is given by

$$\begin{aligned} \langle \hat{n}_4 \rangle &= T^2 \langle \hat{n}_1 \rangle + R^2 \langle \hat{n}_2 \rangle - iRT \langle \hat{a}_2^\dagger \hat{a}_1 - \hat{a}_1^\dagger \hat{a}_2 \rangle \\ &= T^2 \langle \hat{n}_1 \rangle + R^2 |\alpha_L|^2 - iRT |\alpha_L| \langle \hat{a}_1 e^{-i\phi_L} - \hat{a}_1^\dagger e^{i\phi_L} \rangle \end{aligned}$$

Neglecting the small term $T^2 \langle \hat{n}_1 \rangle$ and the signal-independent term $(1 - T^2)|\alpha_L|^2$, the measured signal is proportional to $\langle \hat{X}_{\phi_L + \pi/2} \rangle$ where

$$\hat{X}_\theta = \hat{a} e^{-i\theta} + \hat{a}^\dagger e^{i\theta}$$

is the quadrature operator for the signal beam. By varying the phase of the local oscillator, different quadratures can be measured.

6.2.2 Balanced Homodyne Detection

Balanced Homodyne Detection is usually preferred to eliminate the contribution of the local oscillator. Here a 50:50 beam splitter is used and the measured signal is the photon number difference between the two output ports:

$$n_{34} = \langle \hat{n}_3 - \hat{n}_4 \rangle = -|\alpha_L| \langle \hat{X}_{\phi_L + \pi/2} \rangle$$

The fluctuations of n_{34} then gives the fluctuations of the quadrature operator of the signal field.

Chapter 7

Coherence Properties of the Electromagnetic Field

7.1 Optical Coherence and Statistical Optics

Coherence plays a central role in modern physics. Loosely speaking, a process is coherent if it is characterized by the existence of some well-defined deterministic phase relationship, or in other words, if some phase is not subject to random noise. Coherence is usually characterized by correlation functions. Coherence in time at the same position is called the temporal coherence, while that in space at the same time is called the spatial coherence. For simplicity, in the discussion here, we assume that the electric field has a single linear polarization and hence can be regarded as a scalar.

Natural light sources radiate fields with fluctuating amplitude and phase, hence the electric field possesses statistical uncertainty. Generally we consider quasi-monochromatic and paraxial fields whose amplitude can be written as

$$E^{(+)}(\mathbf{r},t) = \mathcal{E}(\mathbf{r},t)\, e^{i(\mathbf{k}\cdot\mathbf{r}-\omega t)}$$

where $\mathcal{E}(\mathbf{r},t)$ is the slowly-varying envelope satisfying

$$|\nabla\mathcal{E}| \ll k\mathcal{E}, \quad |\dot{\mathcal{E}}| \ll \omega\mathcal{E}$$

Due to statistical fluctuations, $\mathcal{E}$ is a random variable with probability distribution $P(\{\mathcal{E}\},t)$. Measurement of $f(\mathcal{E})$ yields the expectation value

$$\langle f(\mathcal{E})\rangle_t = \int d\{\mathcal{E}\}\, P(\{\mathcal{E}\},t) f(\mathcal{E})$$

Usually we deal with stationary and ergodic sources.

- *Stationarity* — statistics is time-independent, [i.e., $P(\{\mathcal{E}\})$], even though any realization of the ensemble $\mathcal{E}$ changes continually in time. Hence $\langle f(\mathcal{E})\rangle_t = \langle f(\mathcal{E})\rangle_{t+T}$, invariant under time translation.

- *Ergodicity* — time average = ensemble average = expectation value, i.e.,

$$\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} dt\, f(\mathcal{E}) = \langle f(\mathcal{E})\rangle$$

An example of a random light source is the "collisional broadened" atomic emission. N atoms each emits light witth a field $E^{(+)} = E_0 e^{-i\omega t}e^{i\phi(t)}$ where $\phi(t)$ is a phase disrupted by frequent collisions between atoms, hence is random. The total field is then $E_T^{(+)} = E_0 e^{-i\omega t}\alpha(t)$ with $\alpha(t) = \sum_j e^{i\phi_j(t)}$. $\alpha(t)$ behaves like a random walk whose probability distribution is a Gaussian (central limit theorem):

$$P(\alpha(t)) = \frac{1}{\pi N}e^{-|\alpha(t)|^2/N}$$

Random variables obey Gaussian statistics are called Gaussian variates. Gaussian variates are particularly important since according to the central limit theorem, any fluctuating random variable will tend to become a Gaussian variate when the fluctuations contain contributions from a large number of independent causes.

If z_1, z_2, ..., z_N is a set of N Gaussian variates, then

$$\langle \Delta z_{i_1}^* \Delta z_{i_2}^* ... \Delta z_{i_N}^* \Delta z_{j_N} ... \Delta z_{j_1}\rangle = \sum_{\text{all } N! \text{ parings}} \langle \Delta z_{i_1}^* \Delta z_{j_1}\rangle\langle \Delta z_{i_2}^* \Delta z_{j_2}\rangle ... \langle \Delta z_{i_N}^* \Delta z_{j_N}\rangle$$

where $\Delta z_i = z_i - \langle z_i\rangle$ is the variance of z_i. This is called the *Gaussian moment theorem.* It follows from this, that if $\langle z_1\rangle = 0 = \langle z_2\rangle$, then

$$\langle z_1^* z_2^* z_2 z_1\rangle = \langle z_1^* z_1\rangle\langle z_2^* z_2\rangle + \langle z_1^* z_2\rangle\langle z_2^* z_1\rangle$$

7.2 First-Order Correlation

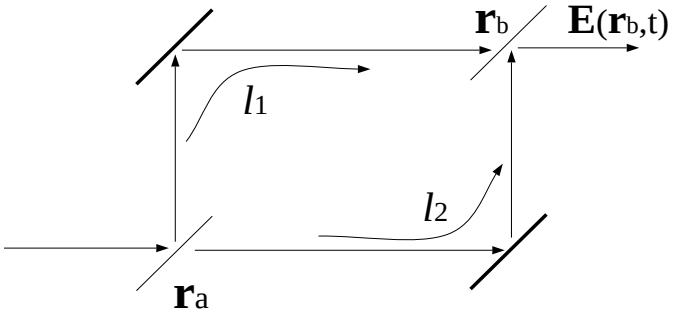

Figure 7-1: Mach-Zehnder interferometer.

The idea of coherence in optics was first associated with the possibility of producing interference fringes when two fields are superposed. Consider the Mach-Zehnder interferometer as show in figure. The field $E(\mathbf{r}_b, t)$ contains two contributions:

$$E^{(+)}(\mathbf{r}_b, t) = E^{(+)}(\mathbf{r}_a, t_1) + E^{(+)}(\mathbf{r}_a, t_2)$$

with $t_i = t - l_i/c$. The output intensity is then

$$I(\mathbf{r}_b, t) = \langle E^{(-)}(\mathbf{r}_b, t) E^{(+)}(\mathbf{r}_b, t) \rangle = \langle |E(x_1)|^2 \rangle + \langle |E(x_2)|^2 \rangle + 2\mathrm{Re}\langle E^{(-)}(x_1) E^{(+)}(x_2) \rangle$$

where $x_i = (\mathbf{r}_a, t_i)$.

Define the *first-order correlation function* as

$$G^{(1)}(x, x') = \langle E^{(-)}(x) E^{(+)}(x') \rangle$$

then

$$I(x) = G^{(1)}(x_1, x_1) + G^{(1)}(x_2, x_2) + 2\mathrm{Re}\left[G^{(1)}(x_1, x_2) \right]$$

The first two terms at the r.h.s. are the intensities from each slit in the absence of the other, while the last term represents the interference term.

Obviously $G^{(1)}(x, x) \geq 0$ and according to the Schwarz inequality,

$$G^{(1)}(x_1, x_1) G^{(1)}(x_2, x_2) \geq |G^{(1)}(x_1, x_2)|^2$$

It is sometimes more convenient to use the normalized correlation function defined as

$$g^{(1)}(x_1, x_2) = \frac{G^{(1)}(x_1, x_2)}{\sqrt{G^{(1)}(x_1, x_1) G^{(1)}(x_2, x_2)}}$$

then we have $|g^{(1)}(x_1, x_2)| \leq 1$. If $|g^{(1)}(x_1, x_2)| = 1$, then the light is first-order coherent; if $|g^{(1)}(x_1, x_2)| = 0$, then the light is incoherent; and if $0 < |g^{(1)}(x_1, x_2)| < 1$, the light is partially coherent. If $G^{(1)}(x_1, x_1) = G^{(1)}(x_2, x_2) = I_0$, and the light field is stationary, i.e., $g^{(1)}(x_1, x_2) = g^{(1)}(\tau)$ with $\tau = t_2 - t_1$, then the output intensity becomes

$$I(x) = 2I_0 \left[1 + \mathrm{Re}[g^{(1)}(\tau)] \right]$$

For quasi-monochromatic field, $g^{(1)}(\tau) = \langle \mathcal{E}^*(\tau) \mathcal{E}(0) \rangle e^{i\omega\tau} / I_0 = |g^{(1)}(\tau)| e^{i\omega\tau + i\phi(\tau)}$. Therefore,

$$I(x) = 2I_0 \left[1 + |g^{(1)}(\tau)| \cos(\omega t + \phi(\tau)) \right]$$

i.e., the envelope of the interference pattern yields $|g^{(1)}(\tau)|$. The visibility of the interference pattern is also $|g^{(1)}(\tau)|$ since

$$\mathcal{V} = \frac{I_{\max} - I_{\min}}{I_{\max} + I_{\min}} = \frac{2I_0(1 + |g^{(1)}(\tau)|) - 2I_0(1 - |g^{(1)}(\tau)|)}{2I_0(1 + |g^{(1)}(\tau)|) + 2I_0(1 - |g^{(1)}(\tau)|)} = |g^{(1)}(\tau)|$$

In general when $G^{(1)}(x_1, x_2)$ factorizes such that

$$G^{(1)}(x_1, x_2) = \mathcal{E}^*(x_1) \mathcal{E}(x_2)$$

where $\mathcal{E}(x)$ is some complex function, not necessarily the electric field, then we say the field is *first-order coherent*. Stationary fields that are first order coherent must be monochromatic, since

$$G^{(1)}(t_1, t_2) = G^{(1)}(t_2 - t_1) = \mathcal{E}^*(t_1) \mathcal{E}(t_2) \Rightarrow \mathcal{E}(t) \sim e^{-i\omega t}$$

Mach-Zehnder interferometer measures correlation functions at the same spatial position but different time, hence it measures the temporal coherence of the field. $G^{(1)}(t, t')$ is sometimes called the *auto-correlation*

function, which gives information about the way a dynamical variable (here the electric field) is changing with respect to its own past history. For stationary light, $G^{(1)}(t,t') = G^{(1)}(\tau) = \langle E^{(-)}(t)E^{(+)}(t+\tau)\rangle$ with $\tau = t' - t$. Now we want to show that the Fourier transformation of $G^{(1)}(\tau)$ is the *spectral density* or *power spectrum* of the field. To this end, we define the Fourier components of the field as

$$\tilde{E}(\omega) = \frac{1}{2\pi}\int dt\, E^{(+)}(t)e^{i\omega t}$$

Then we have

$$\langle \tilde{E}^*(\omega)\tilde{E}(\omega')\rangle = \frac{1}{(2\pi)^2}\int dt \int dt'\, \langle E^{(-)}(t)E^{(+)}(t')\rangle\, e^{i(\omega' t' - \omega t)} = \frac{1}{(2\pi)^2}\int dt\, e^{i(\omega'-\omega)t}\int d\tau\, G^{(1)}(\tau)e^{i\omega'\tau} = S(\omega)\delta(\omega'-\omega)$$

where

$$S(\omega) = \lim_{\Delta\omega\to 0}\int_{\omega-\Delta\omega/2}^{\omega+\Delta\omega/2} d\omega'\, \langle \tilde{E}^*(\omega)\tilde{E}(\omega')\rangle = \frac{1}{2\pi}\int d\tau\, G^{(1)}(\tau)\, e^{i\omega\tau} \tag{7.1}$$

The first equation suggests that $S(\omega)$ is just the spectral density of the field. Eq. (7.1), together with its inverse

$$G^{(1)}(\tau) = \int d\omega\, S(\omega)\, e^{-i\omega\tau}$$

are generally known as the *Wiener-Khintchine theorem*.

7.3 Second-Order Correlation

We can similarly define higher-order correlation functions:

$$G^{(n)}(x_1,...,x_n;y_n,...,y_1) = \langle E^{(-)}(x_1)...E^{(-)}(x_n)E^{(+)}(y_n)...E^{(+)}(y_1)\rangle$$

and the normalized ones

$$g^{(n)}(x_1,...,x_n;y_n,...,y_1) = \frac{G^{(n)}(x_1,...,x_n;y_n,...,y_1)}{\left[\Pi_{r=1}^n G^{(1)}(x_r,x_r)G^{(1)}(y_r,y_r)\right]^{1/2}}$$

These correlation functions obey

$$G^{(n)}(x_1,...,x_n;x_n,...,x_1) \geq 0,$$
$$G^{(n)}(x_1,...,x_n;x_n,...,x_1)G^{(n)}(y_n,...,y_1;y_1,...,y_n) \geq |G^{(n)}(x_1,...,x_n;y_n,...,y_1)|^2$$

A field is said to have mth-order coherence if its nth-order correlation function factorizes as

$$G^{(n)}(x_1,...,x_n;y_n,...,y_1) = \mathcal{E}^*(x_1)...\mathcal{E}^*(x_n)\mathcal{E}(y_n)...\mathcal{E}(y_1)$$

for all $n \leq m$. By this definition, a monochromatic field ($E^{(+)} \sim e^{i\mathbf{k}\cdot\mathbf{r}}$) is coherent to all orders.

The most important one is the second-order correlation function, which measures the correlation of intensity. (First-order correlation measures the correlation of amplitude.) The first experiment performed to make such a measurement is the Hanbury-Brown and Twiss experiment, which measures the quantity

$$G^{(2)}(t_1,t_2) = \langle I(t_1)I(t_2)\rangle = \langle E^{(-)}(t_1)E^{(-)}(t_2)E^{(+)}(t_2)E^{(+)}(t_1)\rangle$$

Usually we deal with *stationary* light fields, that is, the influences that govern the fluctuations do not change with time. Then the average only depends on the difference $\tau = t_2 - t_1$. Hence we write

$$G^{(2)}(\tau) = \langle E^{(-)}(t)E^{(-)}(t+\tau)E^{(+)}(t+\tau)E^{(+)}(t)\rangle = \langle E^{(-)}(0)E^{(-)}(\tau)E^{(+)}(\tau)E^{(+)}(0)\rangle$$

The normalized second-order correlation function is defined as

$$g^{(2)}(\tau) = \frac{G^{(2)}(\tau)}{|G^{(1)}(0)|^2}$$

If the field is second-order coherent, then

$$G^{(1)}(\tau) = \mathcal{E}^*(0)\mathcal{E}(\tau), \quad \text{and} \quad G^{(2)}(\tau) = \mathcal{E}^*(0)\mathcal{E}^*(\tau)\mathcal{E}(\tau)\mathcal{E}(0)$$

which implies that

$$g^{(2)}(\tau) = 1$$

that is, the normalized second-order correlation function $g^{(2)}(\tau)$ is unity and is independent of the time delay τ.

7.3.1 Hanbury Brown-Twiss Interferometer

Suppose our interferometer has very long path lengths. Generally it is very difficult to keep the interferometer balanced since a small fluctuation in path length will lead to a large shift in rapidly oscillating interference pattern. Hanbury Brown and Twiss employed a new technique to measure the coherence. The Hanbury Brown-Twiss interferometer is shown in the figure.

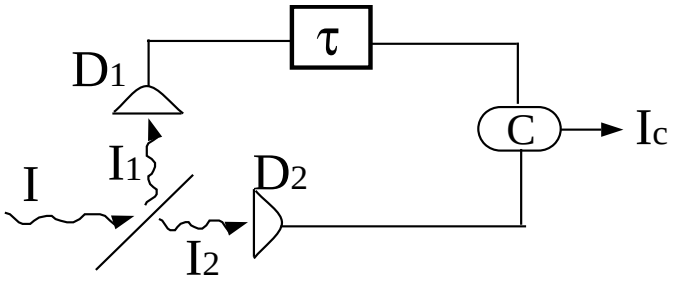

Figure 7-2: Hanbury Brown-Twiss interferometer.

The interferometer measures the second order correlation function $\langle I(t+\tau)I(t)\rangle$. How then can second order intensity correlation function yields first order phase coherence? This is precisely due to the Gaussian moment theorem.

Since the electric field amplitudes are Gaussian variates with zero mean, then we have

$$\begin{aligned}\langle I(t+\tau)I(t)\rangle &= \langle E^{(-)}(t)E^{(-)}(t+\tau)E^{(+)}(t+\tau)E^{(+)}(t)\rangle \\ &= \langle E^{(-)}(t)E^{(+)}(t)\rangle\langle E^{(-)}(t+\tau)E^{(+)}(t+\tau)\rangle + \langle E^{(-)}(t)E^{(+)}(t+\tau)\rangle\langle E^{(-)}(t+\tau)E^{(+)}(t)\rangle \\ &= I_0^2\left[1 + |g^{(1)}(\tau)|^2\right]\end{aligned}$$

where $I_0 = \langle E^{(-)}(t)E^{(+)}(t)\rangle = \langle E^{(-)}(t+\tau)E^{(+)}(t+\tau)\rangle$.

The intensity auto-correlation thus yields the first order correlation $|g^{(1)}(\tau)|$! Technically this is much easier for very long armed interferometers since light intensity is not sensitive to path length fluctuations. Historically HBT first did their experiment to measure spatial coherence of stars in 1950's.

HBT interferometer works not only for photons, but also for other bosonic particles. Nuclear physicists use this technique to determine the size of nuclei.

7.4 Classical vs. Quantum Correlation

The above definitions for correlation functions is valid for both classical and quantum fields, only that for quantum fields, the field amplitudes are replaced by field operators. For classical fields, the order the E's are irrelevant. For quantum fields, we know that the order is important as annihilation and creation operators do not commute, so why should this particular ordering be adopted? The reason lies in the way how the photon is detected. Most detectors detect a photon by absorption, i.e., by annihilating the photon. Consider a photon detector which detects the photon by atomic ionization. Let us consider the rate at which a particular atom in a photon detector is ionized by a beam of photons. Suppose the atom is initially in state $|i\rangle$ and the ionization occurs by absorption of a photon $\mathbf{k}s$. The relevant matrix element of the electric-dipole interaction is

$$\langle f|\hat{\mathbf{D}}|i\rangle \cdot \langle \psi_f|\hat{\mathbf{E}}^{(+)}_{\mathbf{k}s}|\psi_i\rangle$$

where ψ_i and ψ_f are the initial and final states of the field, respectively. The transition rate Γ which measures the photon intensity is proportional to the modular square of the above quantity (Fermi's golden rule). The atomic part of the expression is not needed here, and in most cases, we are not interested in the final state of the field, so the final states are summed over

$$\Gamma \propto \sum_f |\langle \psi_f|\hat{\mathbf{E}}^{(+)}_{\mathbf{k}s}|\psi_i\rangle|^2 = \sum_f \langle \psi_i|\hat{\mathbf{E}}^{(-)}_{\mathbf{k}s}|\psi_f\rangle\langle \psi_f|\hat{\mathbf{E}}^{(+)}_{\mathbf{k}s}|\psi_i\rangle = \langle \psi_i|\hat{\mathbf{E}}^{(-)}_{\mathbf{k}s}\hat{\mathbf{E}}^{(+)}_{\mathbf{k}s}|\psi_i\rangle$$

where the final step is readily verified using $\sum_f |\psi_f\rangle\langle\psi_f| = 1$. Thus we see why this particular ordering is used. Such an ordering with any creation operators to the left of any annihilation operators are called normal ordering. Of course if the detector works by making a stimulated emission, then the anti-normal ordered correlation function should be used. Any normal ordered correlation function factorizes into a product of complex functions when the field is in the coherent state. Therefore coherent state is coherent to all orders.

Although they have similar expressions, properties of correlation functions in the classical and the quantum regimes may differ.

Consider $g^{(2)}(0)$. For classical field, we have

$$G^{(1)}(0) = \langle I\rangle, \quad G^{(2)}(0) = \langle I^2\rangle$$

Hence we have

$$g^{(2)}(0) - 1 = \frac{G^{(2)}(0) - |G^{(1)}(0)|^2}{|G^{(1)}(0)|^2} = \frac{\langle I^2\rangle - \langle I\rangle^2}{\langle I\rangle^2} = \frac{\langle (\Delta I)^2\rangle}{\langle I\rangle^2} \geq 0$$

Therefore for classical field,

$$g^{(2)}(0) \geq 1 \tag{7.2}$$

Furthermore, Let us consider a measurement of intensity correlation of a classical field. We perform N measurements, and obtain values $I(t_1)I(t_1+\tau)$, $I(t_2)I(t_2+\tau)$, ...,$I(t_N)I(t_N+\tau)$. The Cauchy-Schwarz inequality gives

$$\left[\sum_{i=1}^{N} I(t_i)I(t_i+\tau)\right]^2 \leq \left[\sum_{i=1}^{N} I^2(t_i)\right]\left[\sum_{i=1}^{N} I^2(t_i+\tau)\right]$$

For a sufficiently long and numerous series of measurements, the two summations on the right are equal. Then we have

$$\langle I(t)I(t+\tau)\rangle \leq \langle I^2(t)\rangle$$

which implies

$$g^{(2)}(\tau) \leq g^{(2)}(0) \tag{7.3}$$

The field property represented by the above inequality is called *photon bunching* which implies that the photons show a tendency to arrive in bunches since the rate of coincidence at zero delay is larger than the rate at finite delay.

For quantum fields, both (7.2) and (21.3) might be violated. For example, for a single-mode quantum field, we have

$$g^{(2)}(0) = \frac{\langle \hat{a}^\dagger \hat{a}^\dagger \hat{a}\hat{a}\rangle}{\langle \hat{a}^\dagger \hat{a}\rangle^2} = \frac{\langle \hat{n}(\hat{n}-1)\rangle}{\langle \hat{n}\rangle^2} = 1 + \frac{\langle (\Delta\hat{n})^2\rangle - \langle \hat{n}\rangle}{\langle \hat{n}\rangle^2}$$

It follows then

$$g^{(2)}(0) \geq 1 - \frac{1}{\langle \hat{n}\rangle} \quad \text{for} \quad \langle \hat{n}\rangle \geq 1$$

and the lower limit is 0 for $\langle \hat{n}\rangle < 1$. Therefore $g^{(2)}(0)$ can become less than 1 which is not allowed for classical field. Quantum fields without classical analog are called *non-classical fields*. Light whose degree of second-order coherence $g^{(2)}(0)$ lies in the range $[1-1/\langle \hat{n}\rangle, 1)$ is an example of non-classical light. Hence all Fock states are non-classical. $g^{(2)}(0) < 1$ implies that

$$\langle (\Delta\hat{n})^2\rangle < \langle \hat{n}\rangle$$

Field whose statistics satisfies the above inequality is called *sub-Poissonian*. Similarly field with $\langle (\Delta\hat{n})^2\rangle = \langle \hat{n}\rangle$ and $\langle (\Delta\hat{n})^2\rangle > \langle \hat{n}\rangle$ is called *Poissonian* and *super-Poissonian*, respectively. Sub-Poissonian is a quantum property without classical analogy.

Quantum fields violates the inequality (21.3) are also non-classical. The property associated with $g^{(2)}(\tau) > g^{(2)}(0)$ is called *photon anti-bunching*.

Chapter 8

Quantum Effects in Nonlinear Optics

We will encountered situations in which light of one frequency falling on an atomic system gives rise to light of different frequencies. Stronger nonlinear effects arise when one is dealing with a large number of atoms, or a *nonlinear medium*. It is sometimes permissible to ignore the atomic structure and dynamics, and to treat the medium as a continuum. The subject of the interaction of the incident field with the nonlinear medium is known as nonlinear optics.

In general, the electric displacement vector can be written as

$$\mathbf{D}(\mathbf{r},t) = \varepsilon_0 \mathbf{E}(\mathbf{r},t) + \mathbf{P}(\mathbf{r},t)$$

and it is usually possible to expand the polarization $\mathbf{P}$ induced in the medium in a power series of electric field $\mathbf{E}$ as

$$P_i = \chi^{(1)}_{ij} E_j + \chi^{(2)}_{ijk} E_j E_k + \chi^{(3)}_{ijkl} E_j E_k E_l + ...$$

where the summation on repeated indices is assumed. Here $\boldsymbol{\chi}^{(n)}$ is a susceptibility tensor of rank $n+1$ and in general involves n different frequencies. Thus the above equation is often written in the following form:

$$\begin{aligned} P_i(\omega_1) &= \chi^{(1)}_{ij}(\omega_1;\omega_1) E_j(\omega_1) + \chi^{(2)}_{ijk}(\omega_1;\omega_1-\omega_2,\omega_2) E_j(\omega_1-\omega_2) E_k(\omega_2) \\ &\quad + \chi^{(3)}_{ijkl}(\omega_1;\omega_1-\omega_2-\omega_3,\omega_2,\omega_3) E_j(\omega_1-\omega_2-\omega_3) E_k(\omega_2) E_l(\omega_3) + ... \end{aligned}$$

The term in $\chi^{(2)}_{ijk}$ represents the lowest-order nonlinear contribution to the energy.

8.1 Optical Harmonic Generation

Harmonic generation is the oldest and best known example of a process in nonlinear optics. A monochromatic light beam of frequency ω_1 incident on the nonlinear medium generates a field at the harmonic frequency $\omega_2 = 2\omega_1$. This process is described by terms in $\chi^{(2)}_{ijk}$. In the interaction picture, we can write the Hamiltonian as

$$H = hg\left[\hat{a}_2^\dagger \hat{a}_1^2 + h.c.\right] \tag{8.1}$$

Clearly $\hat{n}_1 + 2\hat{n}_2$ is a constant of motion. From (8.1) we can derive the equations of motion

$$\frac{d}{dt}\hat{a}_1 = -2ig\hat{a}_1^\dagger \hat{a}_2, \quad \frac{d}{dt}\hat{a}_2 = -ig\hat{a}_1^2$$

Similarly the second derivative equations

$$\frac{d^2}{dt^2}\hat{a}_1 = 4g^2(\hat{n}_2 - \hat{n}_1/2)\hat{a}_1, \quad \frac{d^2}{dt^2}\hat{a}_2 = -4g^2(\hat{n}_1 + 1/2)\hat{a}_2$$

For sufficiently short time, we can Taylor expand $\hat{a}_{1,2}(t)$ as

$$\begin{aligned}
\hat{a}_1(t) &= \hat{a}_1(0) + t\frac{d}{dt}\hat{a}_1(0) + \frac{t^2}{2}\frac{d^2}{dt^2}\hat{a}_1(0) + ... = \hat{a}_1(0) - 2igt\hat{a}_1^\dagger(0)\hat{a}_2(0) + 2g^2t^2(\hat{n}_2(0) - \hat{n}_1(0)/2)\hat{a}_1(0) + ... \\
\hat{a}_2(t) &= \hat{a}_2(0) + t\frac{d}{dt}\hat{a}_2(0) + \frac{t^2}{2}\frac{d^2}{dt^2}\hat{a}_2(0) + ... = \hat{a}_2(0) - igt\hat{a}_1^2(0) - 2g^2t^2(\hat{n}_1(0) + 1/2)\hat{a}_2(0) + ...
\end{aligned}$$

up to terms of order $(gt)^2$.

Suppose initially the fundamental mode is in a coherent state $|\alpha\rangle$ and the harmonic mode is in vacuum state. The we have

$$\langle \hat{n}_1(t)\rangle = |\alpha|^2 - 2g^2t^2|\alpha|^4 + ..., \quad \langle \hat{n}_2(t)\rangle = g^2t^2|\alpha|^4 + ...$$

and the number fluctuations

$$\langle (\Delta\hat{n}_1)^2\rangle = \langle \hat{n}_1\rangle - 2g^2t^2|\alpha|^4 + ..., \quad \langle (\Delta\hat{n}_2)^2\rangle = \langle \hat{n}_2\rangle + O(g^6t^6)$$

Therefore the statistics of the harmonic photons are close to Poissonian and that of the fundamental mode are sub-Poissonian.

If we define the quadrature of the fundamental mode as

$$\hat{X}_\phi(t) = \hat{a}_1(t)e^{-i\phi} + \hat{a}_1^\dagger(t)e^{i\phi}$$

It is easy to show that

$$\begin{aligned}
\langle \hat{X}_\phi(t)\rangle &= (\alpha e^{-i\phi} + \alpha^* e^{i\phi})(1 - g^2t^2|\alpha|^2) + ... \\
\langle \hat{X}_\phi^2(t)\rangle &= \left[\alpha^2 e^{-2i\phi}(1 - 2g^2t^2|\alpha|^2) + c.c.\right] + 2|\alpha|^2(1 - 2g^2t^2|\alpha|^2) + 1 - \left[g^2t^2\alpha^2 e^{-2i\phi} + c.c.\right] + ...
\end{aligned}$$

so that

$$\langle [\Delta\hat{X}_\phi(t)]^2\rangle = 1 - 2g^2t^2|\alpha|^2\cos 2(\theta - \phi) + ..., \quad (\alpha = |\alpha|e^{i\theta})$$

If we choose $\phi = \theta$, then the dispersion of $\hat{X}_\theta(t)$ is less than unity, i.e., the fundamental mode is squeezed.

We see therefore that the process of second harmonic generation is not describable completely classically, because it is accompanied by the generation of sub-Poissonian statistics and squeezing, both of which are purely quantum mechanical phenomena.

8.2 Parametric Down-Conversion

The process of parametric down-conversion is, in a sense, the inverse of harmonic generation. In the former case, one pump photon incident on the dielectric having $\boldsymbol{\chi}^{(2)}$ nonlinearity breaks up into two new photons of

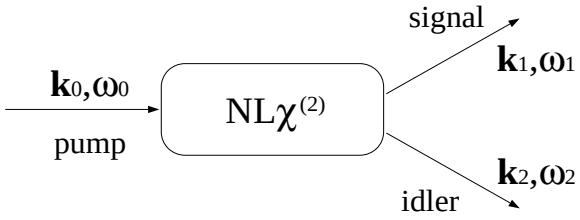

Figure 8-1: Process of parametric down-conversion.

lower frequencies as illustrated in the figure. For historical reasons, the two new photons are known as the *signal photon* and the *idler photon*. If the two new photons are indistinguishable, it is possible to describe the process by the same Hamiltonian (8.1). Here we want to treat the more general situation where the signal and the idler photons are distinguishable.

In the steady state, we have $\omega_0 = \omega_1 + \omega_2$. Phase matching condition also requires that $\mathbf{k}_0 = \mathbf{k}_1 + \mathbf{k}_2$. Experiments showed that the signal and idler photons appeared 'simultaneously' within the resolving time of the detectors and the associated electronics. The Hamiltonian in the interaction picture reads

$$H = \hbar g \left[\hat{a}_1^\dagger \hat{a}_2^\dagger \hat{a}_0 + h.c.\right] \tag{8.2}$$

Clearly $\hat{n}_1 + \hat{n}_2 + 2\hat{n}_0$ is a constant of motion which reflects the fission of one pump photon into one signal and one idler photon. A second constant of motion is $\hat{n}_1 - \hat{n}_2$ which reflects the fact that the signal and the idler photons are always created together. With a similar procedure as above, we can obtain short-time solutions for $\hat{a}_j(t)$ as before. However, under the *undepleted pump* approximation, the problem can be solved analytically.

8.2.1 Solution under the Undepleted Pump Approximation

When the incident pump field is intense and that the pump mode $\hat{a}_0$ can be treated classically as a field of complex amplitude $a_0 = |a_0|e^{i\theta}$. The equations of motion for $\hat{a}_{1,2}$ are then

$$\frac{d}{dt}\hat{a}_1(t) = -iga_0\hat{a}_2(t), \quad \frac{d}{dt}\hat{a}_2(t) = -iga_0\hat{a}_1(t)$$

By differentiating a second time, we can easily solve the above equations as

$$\hat{a}_1(t) = \hat{a}_1(0)\cosh(g|a_0|t) - ie^{i\theta}\hat{a}_2^\dagger(0)\sinh(g|a_0|t) \tag{8.3}$$

$$\hat{a}_2(t) = \hat{a}_2(0)\cosh(g|a_0|t) - ie^{i\theta}\hat{a}_1^\dagger(0)\sinh(g|a_0|t) \tag{8.4}$$

In the Schrödinger picture, if we take the initial states for the signal and the idler mode to be vacuum state, we have

$$|\psi(t)\rangle = U(t,0)|0\rangle = e^{-iHt/\hbar}|0\rangle = \hat{S}_{12}(igta_0)|0\rangle$$

where $\hat{S}_{12}(\xi) = e^{\xi^*\hat{a}_1\hat{a}_2 - \xi\hat{a}_1^\dagger\hat{a}_2^\dagger}$ is the two-mode squeezing operator. When the two modes are identical, we have a degenerate down-conversion, and the resulting propagation operator is just the single-mode squeezing operator.

8.2.2 Photon Statistics

If we take the initial state of both the signal and the idler modes to be the vacuum state, then we can readily calculate the moments of the photon number

$$\langle \hat{n}_1 \rangle = \langle \hat{n}_2 \rangle = \sinh^2(g|a_0|t) \equiv \bar{n}, \quad \langle \hat{n}_1^2 \rangle = \langle \hat{n}_2^2 \rangle = 2\sinh^4(g|a_0|t) + \sinh^2(g|a_0|t) = 2\bar{n}^2 + \bar{n}$$

and we have

$$\langle (\Delta \hat{n}_1)^2 \rangle = \bar{n}(1+\bar{n}) = \langle (\Delta \hat{n}_2)^2 \rangle$$

We can also obtain the cross-correlation as

$$\langle \hat{a}_1^\dagger \hat{a}_2^\dagger \hat{a}_2 \hat{a}_1 \rangle = \sinh^2(g|a_0|t) + 2\sinh^4(g|a_0|t) = 2\bar{n}^2 + \bar{n} \tag{8.5}$$

For $\langle \hat{n}_i(t) \rangle \ll 1$, (8.5) becomes

$$\langle \hat{a}_1^\dagger \hat{a}_2^\dagger \hat{a}_2 \hat{a}_1 \rangle \approx \langle \hat{n}_i(t) \rangle$$

The left hand side is the joint probability of detecting both a signal and an idler photon (by a perfect detector), while the r.h.s. measures the probability that a signal or an idler photon is detected. The above equation therefore shows that the signal and the idler photons are always produced together.

8.2.3 Non-classical Behavior

It is easy to show that

$$\langle \hat{a}_1^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 \rangle = \langle \hat{a}_2^\dagger \hat{a}_2^\dagger \hat{a}_2 \hat{a}_2 \rangle = 2\langle \hat{n}_i(t) \rangle^2 = 2\bar{n}^2$$

Then according to (8.5), we have

$$\langle \hat{a}_1^\dagger \hat{a}_2^\dagger \hat{a}_2 \hat{a}_1 \rangle > \sqrt{\langle \hat{a}_1^\dagger \hat{a}_1^\dagger \hat{a}_1 \hat{a}_1 \rangle \langle \hat{a}_2^\dagger \hat{a}_2^\dagger \hat{a}_2 \hat{a}_2 \rangle}$$

which violates the Schwarz inequality for classical field: $\langle I_1 I_2 \rangle \leq \sqrt{\langle I_1^2 \rangle \langle I_2^2 \rangle}$. Therefore the state produced by the parametric down-conversion has no classical analog, as there are no classical fields with the property that the joint probability of photodetection at two different points in space where the mean intensities are equal is larger than that for two detections at the same point. The Schwartz inequality is strongly violated when $\langle \hat{n}_i(t) \rangle \ll 1$. Experimentally a violation about 600 standard deviations was demonstrated.

The phenomenon of spontaneous parametric down-conversion is capable of providing us with a very close approximation to an ideal one-photon state, that can be used as a probe or input for other processes, by virtue of the fact that when a signal photon is detected, we know that it must be accompanied by a sister idler photon.

8.3 Kerr Medium

A Kerr medium possesses a $\chi^{(3)}$ nonlinear susceptibility. A single-mode field in a Kerr medium is characterized by the interaction Hamiltonian

$$H = \hbar g \hat{a}^{\dagger 2} \hat{a}^2 = \hbar g \left(\hat{n}^2 - \hat{n} \right)$$

Obviously, $\hat{n}$ is a constant of motion.

Suppose that the initial state is a coherent state $|\alpha\rangle$ and for simplicity, assume α is real. Then the state at later time t is given by

$$\begin{aligned} |\psi(t)\rangle &= e^{-iHt/\hbar}|\alpha\rangle = e^{-igt(\hat{n}^2-\hat{n})}e^{-\alpha^2/2}\sum_n \frac{\alpha^n}{\sqrt{n!}}|n\rangle \\ &= e^{-\alpha^2/2}\sum_n \frac{\alpha^n}{\sqrt{n!}}e^{-in(n-1)gt}|n\rangle \end{aligned}$$

In particular, when $gt^* = \pi/2$, using

$$e^{-in(n-1)\pi/2} = (-i)^{n(n-1)} = \frac{1}{\sqrt{2}}\left[i^n e^{-i\pi/4} + (-i)^n e^{i\pi/4}\right]$$

we have

$$|\psi(t^*)\rangle = \frac{1}{\sqrt{2}}e^{-\alpha^2/2}\sum_n \frac{(i\alpha)^n e^{-i\pi/4} + (-i\alpha)^n e^{i\pi/4}}{\sqrt{n!}}|n\rangle = \frac{1}{\sqrt{2}}\left(e^{-i\pi/4}|i\alpha\rangle + e^{i\pi/4}|-i\alpha\rangle\right)$$

Therefore, at this particular time, the state evolves into a superposition of two distinct coherent states. As $|\alpha|$ can be very large, these two states could correspond to two very strong, and hence nearly "classical", light fields. Thus $|\psi(t^*)\rangle$ is like a Schrödinger cat state. The coherent superposition of dinstinct physical states represents the feature of quantum mechanics which most distinguishes it from classical mechanics. Unfortunately this phenomenon is very difficult to observe experimentally as typical values of the coupling coefficient g would require absurdly large interaction time, and also decoherence makes the observation of such a superposition unlikely.

Chapter 9

Bell's Inequality and Quantum Optics

Entanglement lies at the heart of the difference between the quantum and classical multi-particle world. Quantum entanglement has created so much excitement and confusion, has inspired so many deep thoughts and paradoxes. Quantum optics, on the other hand, has been proved to be a wonderful testing ground to elucidate the nature of entanglement.

9.1 Einstein Locality

Consider the atom-photon system introduced in the discussion of density operator. When the photon has not passed thought the filter and is not detected, the state of the system is an entangled state:

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|+1\rangle_a|\sigma_-\rangle_p - |-1\rangle_a|\sigma_+\rangle_p\right)$$

A measurement of one chosen variable of the atom thus completely determines the outcome of the measurement of the corresponding variable of the photon, and vice versa, in spite of the fact that the atom and the photon can in principle be so far apart that no influence resulting from one measurement can possibly propagate to the other particle in the available time.

According to Einstein, Pldolsky and Rosen (EPR) when the outcome of a measurement of some particle variable can be predicted with certainty, without disturbing the particle, then '... there exists an element of physical reality corresponding to this physical quantity...'. In other words, the photon really is in that state, irrespective of whether it is actually measured or not. This must be contrasted with the quantum mechanical point of view — *the measurement creates the reality.*

Now suppose we measure the z-component of the atomic angular moment $(L_z)_a$, the result will predetermine the $(L_z)_p$ of the photon. Then according to EPR, the photon really *has* this value of $(L_z)_p$. But we can also choose to measure $(L_x)_a$, which then predetermines $(L_x)_p$. Hence, according to EPR, the photon has definite values of $(L_x)_p$ and $(L_z)_p$. This is in contradiction of quantum mechanics, since L_x and L_z do not commute, they cannot have definite values at the same time. This contradiction led EPR to conclude that quantum mechanics is incomplete.

The paradox arises because we tend intuitively to think in classical terms, i.e., to associated an objective physical reality with each particle and its variables, whereas in quantum mechanics, a dynamical variable does not actually have a value until it is measured. Attempts have been made to account for the predicted correlations between two particles in terms of *hidden variables*, or unmeasurable parameters that are supposed to determine the outcome of an experiment. But it was later shown by Bell and others that such non-local effects are fundamentally quantum mechanical, and that no realistic local theory can account for the correlations quantitatively. More specifically, Bell constructed the famous Bell's inequality which any local realistic theory must obey. Experiments, however, showed that the Bell's inequality can be violated when an entangled state is used.

9.2 EPR Paradox and an Entangled Two-Photon State

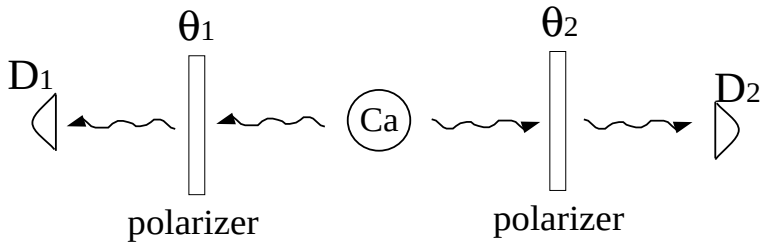

Figure 9-1: Outline of an experiment for testing Bell's inequality.

Consider the atomic cascade discussed in the Introduction. The atom decays from a $J = 0$ excited state to a $J = 0$ ground state via an intermediate state by emitting two photons. Suppose that the photons 1 and 2 leave the atom in opposite directions along the z-axis in the singlet state:

$$|\psi\rangle = \frac{1}{\sqrt{2}} \left(|x_1, y_2\rangle - |y_1, x_2\rangle \right)$$

where $|x_1, y_2\rangle$ means that Photon 1 is x-polarized and Photon 2 is y-polarized. Hence the individual photons are unpolarized but their polarizations are correlated, that is, the two photons have orthogonal polarizations.

Two linear polarizers inclined at angles θ_1 and θ_2 to the x-direction, respectively, are inserted and put before the photodetectors D_1 and D_2. The dynamical variables $\hat{a}_1$ and $\hat{a}_2$ of the field behind the polarizers are related to those $\hat{a}_{1x}$, $\hat{a}_{1y}$, $\hat{a}_{2x}$, $\hat{a}_{2y}$ before the polarizers as

$$\hat{a}_j = \hat{a}_{jx} \cos\theta_j + \hat{a}_{jy} \sin\theta_j$$

Let $P_j(\theta_j)$ be the probability that Photon j is detected (assume perfect detectors), and $P_{12}(\theta_1, \theta_2)$ be the joint probability that both photons are detected by their respective detectors. Then we have

$$P_1(\theta_1) = \langle\psi|\hat{a}_1^\dagger \hat{a}_1|\psi\rangle = \frac{1}{2} = P_2(\theta_2) \tag{9.1}$$

$$P_{12}(\theta_1, \theta_2) = \langle\psi|\hat{a}_1^\dagger \hat{a}_2^\dagger \hat{a}_2 \hat{a}_1|\psi\rangle = \frac{1}{2} \sin^2(\theta_1 - \theta_2) \tag{9.2}$$

(24.6) is a consequence that the individual photons are unpolarized, (9.2) arises from the fact that the two photons have orthogonal polarizations, that is, when $\theta_1 = \theta_2$, the joint probability vanishes; when $\theta_1 - \theta_2 = \pm\pi/2$, P_{12} reaches maximum. The conditional probability $P_c(\theta_2|\theta_1)$ of detecting Photon 2, given the detection of Photon 1, is therefore

$$P_c(\theta_2|\theta_1) = P_{12}(\theta_1, \theta_2)/P_1(\theta_1) = \sin^2(\theta_1 - \theta_2)$$

This again shows that Photon 2 is definitely polarized at right angles to Photon 1; yet the polarization of Photon 1 was chosen at will by the orientation of Polarizer 1. The outcome of the measurement on Photon 2 therefore appears to be influenced by the orientation of the Polarizer 1, even though the two photons may be well separated at the time of measurement. This is the paradox.

If '+' denotes the detection of the photon and '−' denotes failure to detect, then, since the latter probability can be obtained from the former by incrementing θ_1, θ_2 by $\pi/2$, we have

$$P(+, \theta_2; +, \theta_1) = P_{12}(\theta_1, \theta_2) = \frac{1}{2}\sin^2(\theta_1 - \theta_2) = P(-, \theta_2; -, \theta_1)$$

Similarly, the joint probability of only one photon being detected is

$$P(+, \theta_2; -, \theta_1) = \frac{1}{2}\cos^2(\theta_1 - \theta_2) = P(-, \theta_2; +, \theta_1)$$

We obtain the probability $P(+, \theta_2|\theta_1)$ that Photon 2 is detected when Polarizer 1 is set to θ_1 by adding $P(+, \theta_2; +, \theta_1)$ and $P(+, \theta_2; -, \theta_1)$, which gives

$$P(+, \theta_2|\theta_1) = P(+, \theta_2; +, \theta_1) + P(+, \theta_2; -, \theta_1) = \frac{1}{2}$$

which is independent of θ_1, showing that setting the angle of Polarizer 1 has no influence on the single photon detection probability of Photon 2.

9.3 Bell's Inequality

Supposed that a statistical correlation of $A(a)$ and $B(b)$ is due to information carried by and localized within each particle, and that at some time in the past the particles constituting one pair were in contact and communication regarding this information. The information is part of the content of a set of hidden variables, denoted collectively by λ. The probability distribution function of λ is denoted by $\rho(\lambda)$ which satisfies the normalization condition

$$\int_\Gamma \rho(\lambda)\, d\lambda = 1 \tag{9.3}$$

where Γ is the tatal λ space. The results of the two selections are then to be deterministic functions $A(a, \lambda)$ and $B(b, \lambda)$ which can take values ± 1. Locality reasonably requires $A(a, \lambda)$ to be independent of the parameter b and $B(b, \lambda)$ to be likewise independent of a.

Defining the correlation function

$$C(a, b) \equiv \langle A(a)B(b)\rangle = \int_\Gamma A(a, \lambda)B(b, \lambda)\rho(\lambda)\, d\lambda \tag{9.4}$$

Then since $|A(a,\lambda)| = 1$, we have

$$\begin{aligned} |C(a,b) - C(a,b')| &\leq \int_\Gamma |A(a,\lambda)[B(b,\lambda) - B(b',\lambda)]|\rho(\lambda)\, d\lambda = \int_\Gamma |B(b,\lambda) - B(b',\lambda)|\rho(\lambda)\, d\lambda \\ |C(a',b) + C(a',b')| &\leq \int_\Gamma |A(a',\lambda)[B(b,\lambda) + B(b',\lambda)]|\rho(\lambda)\, d\lambda = \int_\Gamma |B(b,\lambda) + B(b',\lambda)|\rho(\lambda)\, d\lambda \end{aligned}$$

Adding these two together, we have

$$|C(a,b) - C(a,b')| + |C(a',b) + C(a',b')| \leq \int_\Gamma [|B(b,\lambda) - B(b',\lambda)| + |B(b,\lambda) + B(b',\lambda)|]\rho(\lambda)\, d\lambda \qquad (9.5)$$

But because each B takes on only the values ± 1, it follows that

$$|B(b,\lambda) - B(b',\lambda)| + |B(b,\lambda) + B(b',\lambda)| = 2$$

and when this results is used in (9.5), we arrive at the *Bell's Inequality*

$$|C(a,b) - C(a,b')| + |C(a',b) + C(a',b')| \leq 2 \qquad (9.6)$$

Let us now apply the inequality to our photon example. Identify a with the polarizer angle θ_1, and b with θ_2, and associate the outcome $+1$ with the detection of the photon, and -1 with non-detection. Then we have

$$C(\theta_1,\theta_2) = P(+,\theta_2;+,\theta_2)+P(-,\theta_2;-,\theta_2)-P(-,\theta_2;+,\theta_2)-P(+,\theta_2;-,\theta_2) = \sin^2(\theta_1-\theta_2)-\cos^2(\theta_1-\theta_2) = -\cos 2(\theta_1-\theta_2$$

With the following particular choice of angles

$$\theta_1 = 0, \;\; \theta_2 = 3\pi/8, \;\; \theta_1' = -\pi/4, \;\; \theta_2' = \pi/8$$

we obtain the result

$$|C(\theta_1,\theta_2) - C(\theta_1,\theta_2')| + |C(\theta_1',\theta_2) + C(\theta_1',\theta_2')| = 2\sqrt{2} > 2$$

which violates Bell's inequality.

9.4 Experimental Confirmation

The above derivation has assumed perfect photodetectors, i.e., detectors with 100% efficiency. Had the quantum efficiencies been incorporated, then Bell's inequality would not have been violated for low efficiencies. To overcome this, other types of Bell's inequalities have been derived that do not directly depend on the detector efficiencies. The experiments were carried out to confirm the violation of these types of inequalities.

The first such experiment that showed a clear violation of Bell's inequality was performed in 1982 by Aspect *et al.* using the atomic cascade of calcium. They found that the inequality is violated by 6 standard deviations. Later experiments showed even stronger violations. Besides the atomic cascade system, the parametric down conversion is also used to create the entangled two-photon state.

Chapter 10

Atom-Field Interaction

A full self-consistent quantum mechanical treatment of atoms interacting with fields requires that we treat the field as well as the matter quantum mechanically. In many cases we can use a simpler theory called *semiclassical theory* in which the atoms are treated quantum mechanically, while the fields are taken to be c-number solutions of the classical Maxwell equations. We'll see both types. But first we need to construct the interaction Hamiltonian. We'll first treat the matter as a charged particle, then extend to the case of an atom.

10.1 Interaction between Field and a Charged Particle

Consider a particle carrying charge e, with mass m. The field is characterized by the vector potential $\mathbf{A}(\mathbf{r},t)$ and the scalar potential $\phi(\mathbf{r},t)$ with corresponding electric and magnetic fields $\mathbf{E}(\mathbf{r},t)$ and $\mathbf{B}(\mathbf{r},t)$.

10.1.1 Classical Hamiltonian

Take $\mathbf{r}$ and $\mathbf{p}$ to be the coordinate and the canonical momentum of the particle. The classical Hamiltonian describing its interaction with the field is

$$H = \frac{1}{2m}|\mathbf{p} - e\mathbf{A}(\mathbf{r},t)|^2 + e\phi(\mathbf{r},t) \tag{10.1}$$

This form of the charge-field Hamiltonian is called the *minimal coupling Hamiltonian.* It is justified in the same way that any Hamiltonian is justified: It leads to the proper equations of motion, as can be shown in the following:

From (10.1), we have

$$\dot{x}_i = \frac{\partial H}{\partial p_i} = \frac{p_i}{m} - \frac{e}{m}A_i(\mathbf{r},t)$$

which leads to the Lagrangian

$$L(\{x_i,\dot{x}_i\},t) = \sum_i p_i\dot{x}_i - H(\{p_i,x_i\},t) = \sum_i \left(\frac{1}{2}m\dot{x}_i^2 + e\dot{x}_iA_i\right) - e\phi$$

from which the Lagrange's equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{x}_i} - \frac{\partial L}{\partial x_i} = 0$$

yields the equation of motion

$$m\ddot{\mathbf{r}} = e\mathbf{E}(\mathbf{r},t) + e\dot{\mathbf{r}} \times \mathbf{B}(\mathbf{r},t)$$

which is exactly the Lorentz force law.

10.1.2 Quantum Mechanical Hamiltonian

The quantum Hamiltonian takes the same form as (10.1), only that the dynamical variables are now quantum operators. There is a deeper reason why the quantum Hamiltonian takes such a form: it is a consequence of *gauge invariance*.

Global gauge invariance is rather trivial in quantum mechanics. If the wave function $\psi(\mathbf{r},t)$ satisfies the Schrödinger equation, then so does $\psi'(\mathbf{r},t) = e^{i\theta}\psi(\mathbf{r},t)$ with θ being a constant independent of position and time.

Things are different for a local gauge transformed wave function. Suppose $\psi(\mathbf{r},t)$ satisfies the Schrödinger equation with the minimal coupling Hamiltonian

$$i\hbar\frac{\partial}{\partial t}\psi(\mathbf{r},t) = \left[\frac{1}{2m}\left(\hat{\mathbf{p}} - e\mathbf{A}(\mathbf{r},t)\right)^2 + e\phi(\mathbf{r},t)\right]\psi(\mathbf{r},t) \tag{10.2}$$

then it can be shown that the wave function

$$\psi'(\mathbf{r},t) \equiv e^{ie\Lambda(\mathbf{r},t)/\hbar}\,\psi(\mathbf{r},t)$$

satisfies the more complicated Schrödinger equation:

$$i\hbar\frac{\partial}{\partial t}\psi'(\mathbf{r},t) = \left[\frac{1}{2m}\left(\hat{\mathbf{p}} - e\mathbf{A}(\mathbf{r},t) - e\nabla\Lambda(\mathbf{r},t)\right)^2 + e\left(\phi(\mathbf{r},t) - \frac{\partial\Lambda(\mathbf{r},t)}{\partial t}\right)\right]\psi'(\mathbf{r},t) \tag{10.3}$$

It is well known that given $\mathbf{A}$ and ϕ, $\mathbf{E}$ and $\mathbf{B}$ are uniquely determined, but not vice versa. This is because the fields are physical quantities, while the potentials are mathematical tools. Different sets of $(\mathbf{A},\phi)$ that lead to the same $(\mathbf{E},\mathbf{B})$ are connected by the *gauge transformation*

$$\mathbf{A}' = \mathbf{A} + \nabla\Lambda, \quad \phi' = \phi - \frac{\partial\Lambda}{\partial t}$$

Therefore we can rewrite (10.3) as

$$i\hbar\frac{\partial}{\partial t}\psi'(\mathbf{r},t) = \left[\frac{1}{2m}\left(\hat{\mathbf{p}} - e\mathbf{A}'(\mathbf{r},t)\right)^2 + e\phi'(\mathbf{r},t)\right]\psi'(\mathbf{r},t)$$

which is now formally identical to (26.5). Therefore *the minimal coupling Hamiltonian is invariant under the local gauge transformation.* Furthermore, under the Coulomb gauge, we have

$$[\hat{\mathbf{p}}, \mathbf{A}] = -i\hbar\nabla\cdot\mathbf{A} = 0$$

10.2 Interaction between an Atom and the Field

For simplicity we take a hydrogen atom with a proton of mass m_p at position $\mathbf{r}_p$ and an electron of mass m_e at $\mathbf{r}_e$. The total Hamiltonian of the atom in the field reads

$$\begin{aligned} H &= H_e + H_p + V(\mathbf{r}_e - \mathbf{r}_p) \\ H_e &= \frac{\mathbf{p}_e^2}{2m_e} - \frac{e}{m_e}\mathbf{A}(\mathbf{r}_e,t)\cdot\mathbf{p}_e + \frac{e^2}{2m_e}\mathbf{A}^2(\mathbf{r}_e,t) \\ H_p &= \frac{\mathbf{p}_p^2}{2m_p} + \frac{e}{m_p}\mathbf{A}(\mathbf{r}_p,t)\cdot\mathbf{p}_p + \frac{e^2}{2m_p}\mathbf{A}^2(\mathbf{r}_p,t) \\ V(\mathbf{r}) &= -\frac{1}{4\pi\varepsilon_0}\frac{e^2}{|\mathbf{r}|^2} \end{aligned}$$

Introducing center-of-mass coordinates and momentum:

$$\mathbf{R} = \frac{m_e\mathbf{r}_e + m_p\mathbf{r}_p}{m_p + m_e} = \frac{m_e}{M}\mathbf{r}_e + \frac{m_p}{M}\mathbf{r}_p, \quad \mathbf{P} = \mathbf{p}_e + \mathbf{p}_p$$

and relative coordinates and momentum:

$$\mathbf{r} = \mathbf{r}_e - \mathbf{r}_p, \quad \mathbf{p} = \frac{m_p}{M}\mathbf{p}_e - \frac{m_e}{M}\mathbf{p}_p = \frac{\mu}{m_e}\mathbf{p}_e - \frac{\mu}{m_p}\mathbf{p}_p$$

where $M = m_e + m_p$ and $\mu = m_e m_p/M$. It's easy to see that

$$\frac{\mathbf{p}_e^2}{2m_e} + \frac{\mathbf{p}_p^2}{2m_p} = \frac{\mathbf{P}^2}{2M} + \frac{\mathbf{p}^2}{2\mu}$$

The vector potential changes on a length scale determined by the wavelength of the field, λ. When λ is much larger than the size of the atom, the electron and the proton both feel essentially the same vector potential, namely the one at the center-of-mass coordinate, i.e.,

$$\mathbf{A}(\mathbf{r}_e,t) \cong \mathbf{A}(\mathbf{r}_p,t) \cong \mathbf{A}(\mathbf{R},t) \tag{10.4}$$

This approximation is called the *dipole approximation*. Under the dipole approximation, we have

$$\begin{aligned} H_e &\cong \frac{\mathbf{p}_e^2}{2m_e} - \frac{e}{m_e}\mathbf{A}(\mathbf{R},t)\cdot\mathbf{p}_e + \frac{e^2}{2m_e}\mathbf{A}^2(\mathbf{R},t) \\ H_p &\cong \frac{\mathbf{p}_p^2}{2m_p} + \frac{e}{m_p}\mathbf{A}(\mathbf{R},t)\cdot\mathbf{p}_p + \frac{e^2}{2m_p}\mathbf{A}^2(\mathbf{R},t) \end{aligned}$$

which result in the following form of the total Hamiltonian:

$$H = \frac{\mathbf{P}^2}{2M} + \frac{\mathbf{p}^2}{2\mu} - \frac{1}{4\pi\varepsilon_0}\frac{e^2}{|\mathbf{r}|^2} - \frac{e}{\mu}\mathbf{A}(\mathbf{R},t)\cdot\mathbf{p} + \frac{e^2}{2\mu}\mathbf{A}^2(\mathbf{R},t) \tag{10.5}$$

This type of expression is sometimes called the $\mathbf{A}\cdot\mathbf{p}$-interaction.

An alternative way to express the Hamiltonian, the so called $\mathbf{d}\cdot\mathbf{E}$-interaction can be derived as follows. An atom consisting of an electron and proton separated by $\mathbf{r}$ possesses a dipole moment $\mathbf{d} = e\mathbf{r}$. Under the dipole approximation, the electric field $\mathbf{E}$ at the position of the electron is essentially identical to that at the position of the proton, i.e.,

$$\mathbf{E}(\mathbf{r}_e,t) \cong \mathbf{E}(\mathbf{r}_p,t) \cong \mathbf{E}(\mathbf{R},t)$$

The dipole in the electric field $\mathbf{E}$ experiences a potential energy

$$H_{\mathbf{r}\cdot\mathbf{E}} = -\mathbf{d}\cdot\mathbf{E}(\mathbf{R},t) = -e\mathbf{r}\cdot\mathbf{E}(\mathbf{R},t)$$

which suggests that the total Hamiltonian of the atom in the field in dipole approximation reads

$$H = \frac{\mathbf{P}^2}{2M} + \frac{\mathbf{p}^2}{2\mu} - \frac{1}{4\pi\varepsilon_0}\frac{e^2}{|\mathbf{r}|^2} - \mathbf{d}\cdot\mathbf{E}(\mathbf{R},t) \tag{10.6}$$

Under the dipole approximation, together with the assumption that the center-of-mass $\mathbf{R}$ is not time dependent, i.e., when the first term at the r.h.s. can be neglected, (10.5) and (10.6) are equivalent. This can be shown by the following manner: If the wave function $\psi(\mathbf{r},t)$ satisfies the time-dependent Schrödinger equation given by Hamiltonian (10.5), then the wave function

$$\psi'(\mathbf{r},t) = e^{-ie\mathbf{r}\cdot\mathbf{A}(\mathbf{R},t)/\hbar}\psi(\mathbf{r},t)$$

satisfies the corresponding Schrödinger equation given by Hamiltonian (10.6). Hence (10.5) and (10.6) are connected via a gauge transformation. For many applications, the $\mathbf{d}\cdot\mathbf{E}$-interaction suffices and is more convenient to use.

The equivalence of the $\mathbf{A}\cdot\mathbf{p}$-interaction and the $\mathbf{d}\cdot\mathbf{E}$-interaction under the dipole approximation is the reason why the dipole approximation is called so. Dipole approximation breaks down when the temperature reaches the single-photon recoil limit. Under such condition, the thermal de Broglie wavelength of the atom becomes comparable with the optical wavelength, the atom can no longer be regarded as a point particle. A proper treatment of this situation requires quantization of the atomic motion.

The Hamiltonian in (10.6) contains several parts. The first part is the center-of-mass kinetic energy $\mathbf{P}^2/(2M)$. We usually assume that the interaction between the atom and the field do not affect the center-of-mass motion of the atom significantly, or we assume that the center-of-mass motion and the internal dynamics of the atom occur in completely different time scales. In either case, this term can be neglected. The part $\mathbf{p}^2/(2\mu) - e^2/(4\pi\varepsilon_0|\mathbf{r}|^2)$ describes the motion of the electron relative to the proton under the Coulomb interaction. This gives rise to atomic level structures that we assume is already known. In quantum optics, the field strength of interest in usually many orders of magnitude weaker than the internal Coulomb electric field of the atom. Hence the interaction won't change the atomic level structures in any significant way. The only part we will be focusing on is thus the interaction term $-\mathbf{d}\cdot\mathbf{E}(\mathbf{R},t)$.

Chapter 11

Jaynes-Cummings Model

The JC model plays an important role in atom-field interaction. It is a fully quantized model and yet analytically solvable. It describes the interaction between a two-level atom and a quantized field.

11.1 Two-Level Atom

The level structures of a real atom look anything but two-level. So how can a two-level-atom (TLA) be a good approximation? The reason lies in two factors: 1) Resonance excitation and 2) Selection rules.

The absorption cross section of an atom absorbing an off-resonant photon is generally of the order of $1\mathring{A}^2$. But when the frequency of the photon matches with the transition frequency from the initial state to some final state, the cross section can be enhanced by many orders of magnitude. This is why the intensities of the lasers used in labs are much less than that required to produce an electric field with one atomic unit ($8.3 \times 10^{16}\mathrm{W/cm}^2$).

Under the resonance condition, many levels lying far away from the resonance can be simply ignored. In addition, the dipole section rules dictates only certain magnetic sublevels are excited. In most cases, the field therefore only causes transitions between a small number of discrete states, in the simplest of which only two states are involved.

The states for a two-level atom: $|g\rangle$ and $|e\rangle$. Their energies are separated by $\hbar\omega_0$, the atomic transition energy. The two states are assumed to have opposite parity (hence dipole transition is allowed) and orthogonal to each other. From these one can construct four independent operators:

$$|g\rangle\langle g|, \;\; |g\rangle\langle e|, \;\; |e\rangle\langle g|, \;\; |e\rangle\langle e|,$$

which form a complete basis. Any arbitrary operator, $\hat{O}$, can then be expanded onto this basis as

$$\hat{O} = O_{gg}\hat{\sigma}_{gg} + O_{ge}\hat{\sigma}_{ge} + O_{eg}\hat{\sigma}_{eg} + O_{ee}\hat{\sigma}_{ee}$$

where $\hat{\sigma}_{ij} = |i\rangle\langle j|$, and $O_{ij} = \langle i|\hat{O}|j\rangle$. In particular, the dipole operator $\hat{\mathbf{d}} = e\hat{\mathbf{r}}$ can be expressed as

$$\hat{\mathbf{d}} = \mathbf{d}_{ge}\hat{\sigma}_{ge} + \mathbf{d}_{eg}\hat{\sigma}_{eg}$$

where we have used the property that states $|g\rangle$ and $|e\rangle$ have opposite parity such that $\langle g|\hat{\mathbf{r}}|g\rangle = \langle e|\hat{\mathbf{r}}|e\rangle = 0$.

11.2 Hamiltonian and the Rotating Wave Approximation

The total Hamiltonian, neglecting the center-of-mass atomic motion, describing the interaction between the two-level and a single mode quantized field reads:

$$H = H_A + H_{\rm int} + H_F$$

where

$$\begin{aligned} H_A &= \hbar\omega_0\hat{\sigma}_{ee}, \quad H_F = \hbar\omega\hat{a}^\dagger\hat{a} \\ H_{\rm int} &= -\hat{\mathbf{d}}\cdot\hat{\mathbf{E}} = -(\mathbf{d}_{ge}\hat{\sigma}_{ge} + \mathbf{d}_{eg}\hat{\sigma}_{eg})\cdot\sqrt{\frac{\hbar\omega}{2\varepsilon_0 V}}\,[\mathbf{u}(\mathbf{R})\,\hat{a} + h.c.] = -\hbar g(\hat{\sigma}_{ge}+\hat{\sigma}_{eg})(\hat{a}+\hat{a}^\dagger) \end{aligned}$$

where $\mathbf{u}(\mathbf{R})$ is the mode function of the field at the center-of-mass coordinate $\mathbf{R}$, and we have assumed $\hbar g = \mathbf{d}_{ge}\cdot\mathbf{u}(\mathbf{R})\sqrt{\hbar\omega/(2\varepsilon_0 V)}$ to be real by a proper choice of phase.

Let us now concentrate on the interaction Hamiltonian

$$H_{\rm int} = -\hbar g(\hat{\sigma}_{ge}+\hat{\sigma}_{eg})(\hat{a}+\hat{a}^\dagger) = -\hbar g(\hat{\sigma}_{eg}\hat{a} + \hat{\sigma}_{ge}\hat{a}^\dagger + \hat{\sigma}_{ge}\hat{a} + \hat{\sigma}_{eg}\hat{a}^\dagger)$$

which contains 4 terms. To further simplify this, we invoke the *rotating wave approximation* and neglect the last two terms. There are two ways to understand the RWA.

11.2.1 Violation of Energy Conservation

The terms $\hat{\sigma}_{ge}\hat{a}$ and $\hat{\sigma}_{eg}\hat{a}^\dagger$ violate energy conservation. The former de-excites the atom and simultaneously absorbs a photon, and the latter excites the atom while it emits a photon. By contrast, the two terms we have kept conserve energy.

11.2.2 Method of Averaging

The above argument is rather heuristic. We can make it more rigorous. For this purpose, it is easiest to work in the interaction picture. Decompose the total Hamiltonian as $H = H_0 + H_{\rm int}$ where $H_0 = H_A + H_F$. In the interaction picture, we have

$$H_{\rm int}^{(I)} = e^{iH_0 t/\hbar}H_{\rm int}e^{-iH_0 t/\hbar} = -\hbar g e^{iH_A t/\hbar}(\hat{\sigma}_{ge}+\hat{\sigma}_{eg})e^{-iH_A t/\hbar}\,e^{iH_F t/\hbar}(\hat{a}+\hat{a}^\dagger)e^{-iH_F t/\hbar}$$

It's easy to show that

$$e^{iH_A t/\hbar}\hat{\sigma}_{ge}e^{-iH_A t/\hbar} = e^{-i\omega_0 t}\hat{\sigma}_{ge}, \;\; e^{iH_A t/\hbar}\hat{\sigma}_{eg}e^{-iH_A t/\hbar} = e^{i\omega_0 t}\hat{\sigma}_{eg}$$

$$e^{iH_F t/\hbar}\hat{a}e^{-iH_F t/\hbar} = e^{-i\omega t}\hat{a}, \;\; e^{iH_F t/\hbar}\hat{a}^\dagger e^{-iH_F t/\hbar} = e^{i\omega t}\hat{a}^\dagger$$

Therefore

$$H_{\rm int}^{(I)} = -\hbar g(e^{-i\omega_0 t}\hat{\sigma}_{ge} + e^{i\omega_0 t}\hat{\sigma}_{eg})(e^{-i\omega t}\hat{a} + e^{i\omega t}\hat{a}^\dagger)$$

We note that the terms $\hat{\sigma}_{ge}\hat{a}$ and $\hat{\sigma}_{eg}\hat{a}^\dagger$ that do not conserve energy are multiplied by oscillatory terms which involve the sum of the frequencies of the field and the atomic transition, while terms $\hat{\sigma}_{eg}\hat{a}$ and $\hat{\sigma}_{ge}\hat{a}^\dagger$ which

do conserve energy are multiplied by terms involving the difference of the two frequencies. For the near resonant case we are most interested in, $|\omega - \omega_0| \ll \omega + \omega_0$.

Since the Schrödinger equation is a differential equation of first order in time we have to integrate in time. This time integration brings the frequency sum and difference into the denominator. Hence the dominant conribution must come from the slowly varying part. The interaction Hamiltonian in the interaction picture is then given by

$$H_{\text{int}}^{(I)} = -\hbar g(\hat{\sigma}_{ge}\hat{a}^\dagger\, e^{i\Delta t} + \hat{\sigma}_{eg}\hat{a}\, e^{-i\Delta t})$$

with $\Delta = \omega - \omega_0$. This corresponding in the original Schrödinger picture to the interaction Hamiltonian

$$H_{\text{int}} = -\hbar g(\hat{\sigma}_{ge}\hat{a}^\dagger + \hat{\sigma}_{eg}\hat{a})$$

The neglected counter-rotating terms would cause a frequency shift of the atomic levels on the order of g^2/ω_0, this is called the *Bloch-Siegert Shift*.

11.3 Dressed States of the JC Model

Let us now go back to the Schrödinger picture under the RWA. The Hamiltonian reads

$$H = \hbar\omega_0\hat{\sigma}_{ee} + \hbar\omega\hat{a}^\dagger\hat{a} - \hbar g(\hat{\sigma}_{eg}\hat{a} + \hat{\sigma}_{ge}\hat{a}^\dagger) \tag{11.1}$$

This Hamiltonian possesses a constant of motion:

$$N_{\text{ex}} = \langle\hat{a}^\dagger\hat{a}\rangle + \langle\hat{\sigma}_{ee}\rangle$$

which can be understood as the *excitation number* of the system. We use Fock states as our basis for the field, then the state with $N_{\text{ex}} = n > 0$ contains two and only two states: $|n-1; e\rangle$ and $|n; g\rangle$. Therefore the Hamiltonian matrix can be decomposed into blocks of 2×2 matrices, each block corresponding to a particular N_{ex}.

States $|n-1; e\rangle$ and $|n; g\rangle$ are eigenstates of $H_A + H_F$, i.e., the uncoupled system. For strong atom-field coupling, they are no longer accurate description of the combined atom-field system. Instead, we should try to find the eigenvalues and eigenstates of the entire Hamiltonian. In most cases this is a impossible task. The beauty of the JC model is that non-trivial but exact solutions to the full Hamiltonian can be obtained in the RWA. This is precisely due to the fact that we can decompose the Hamiltonian matrix into 2×2 sub-blocks.

Let us focus on the nth block spanned by $|n-1; e\rangle$ and $|n; g\rangle$. The submatrix can be easily obtained as

$$\hbar\begin{pmatrix} (n-1)\omega + \omega_0 & -\sqrt{n}g \\ -\sqrt{n}g & n\omega \end{pmatrix} = \hbar\left(n\omega - \frac{\Delta}{2}\right)\mathbf{1} + \frac{\hbar}{2}\begin{pmatrix} -\Delta & -R_n \\ -R_n & \Delta \end{pmatrix}$$

where $\Delta = \omega - \omega_0$ is the laser detuning and $R_n = 2\sqrt{n}g$ is the Rabi frequency for the nth block. The eigenvalues can be easily found

$$E_{n\pm} = \hbar\left(n\omega - \frac{\Delta}{2}\right) \pm \frac{\hbar}{2}\sqrt{\Delta^2 + R_n^2}$$

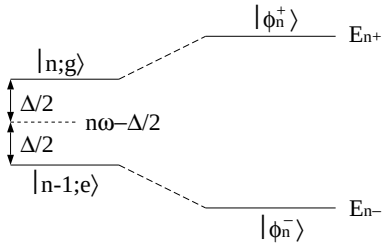

Figure 11-1: Dressed states of the JC model.

with corresponding eigenstates

$$|\phi_n^+\rangle = -\cos\frac{\theta_n}{2}|n-1;e\rangle + \sin\frac{\theta_n}{2}|n;g\rangle \tag{11.2}$$

$$|\phi_n^-\rangle = \sin\frac{\theta_n}{2}|n-1;e\rangle + \cos\frac{\theta_n}{2}|n;g\rangle \tag{11.3}$$

where

$$\cos\frac{\theta_n}{2} = \sqrt{\frac{\Omega_n - \Delta}{2\Omega_n}}, \quad \sin\frac{\theta_n}{2} = \sqrt{\frac{\Omega_n + \Delta}{2\Omega_n}}$$

with $\Omega_n = \sqrt{\Delta^2 + R_n^2} = \sqrt{\Delta^2 + 4ng^2}$ being the generalized Rabi frequency. The auxilliary angle θ_n can also be defined as $\theta_n = \tan^{-1}(-R_n/\Delta)$.

11.4 Entanglement and Vacuum Rabi Oscillation

The eigenstates of the JC Hamiltonian $|\phi_n^\pm\rangle$ are called *dressed states*, they are also nonseparable *entangled* states, i.e., states that cannot be factored into a product of an atom state and a field state. Entanglement can be a useful property in cavity QED experiment. For example, consider the initial atom-field state consists an excited atom outside the empty cavity, i.e., $|\Psi(0)\rangle = |0;e\rangle$. After the atom passing through the cavity, the state evolves into $|\Psi(T)\rangle = \alpha|0;e\rangle + \beta|1;g\rangle$. In this way the atom can be used as a "pointer". That is, the observation of the atom's quantum state (whether in $|e\rangle$ or $|g\rangle$) after it has left the cavity yields a number that perfectly correlates with the quantum state of the field, without in any way "touching" the cavity field. This is an example of what is called a QND (quantum non-demolition) measurement.

The above example shows that the initial state $|\Psi(0)\rangle = |0;e\rangle$ is not stationary for the coupled system. We can make it more quantitatively to study the dynamics of this state. To this end, let us find the inverse relations of (12-1) and (12.2):

$$|n-1;e\rangle = -\cos\frac{\theta_n}{2}|\phi_n^+\rangle + \sin\frac{\theta_n}{2}|\phi_n^-\rangle \tag{11.4}$$

$$|n;g\rangle = \sin\frac{\theta_n}{2}|\phi_n^+\rangle + \cos\frac{\theta_n}{2}|\phi_n^-\rangle \tag{11.5}$$

The state $|\Psi(0)\rangle = |0;e\rangle$ will then evolve in time as

$$|\Psi(t)\rangle = e^{-i(\omega-\Delta/2)t}\left(-\cos\frac{\theta_1}{2}e^{-i\Omega_1 t/2}|\phi_1^+\rangle + \sin\frac{\theta_1}{2}e^{i\Omega_1 t/2}|\phi_n^-\rangle\right)$$

The probability that the system flips from $|0;e\rangle$ to $|1;g\rangle$, i.e., the probability that a photon will be emitted into an empty cavity, is given by

$$p = |\langle 1;g|\Psi(t)\rangle|^2 = \sin^2\theta_1 \sin^2\frac{\Omega_1 t}{2} = \frac{R_1^2}{\Omega_1^2}\sin^2\frac{\Omega_1 t}{2}$$

Spontaneous emission in free space produces monotonic and irreversible decay of upper-level amplitude, whereas here we find the so-called *vacuum Rabi oscillation*.

11.5 Coherent Field State: Collapse and Revival

Let us now consider the field is initially in a coherent state $|\alpha\rangle$ while the atom is initially in its ground state $|g\rangle$, i.e., $|\Psi(0)\rangle = |\alpha\rangle \otimes |g\rangle$ which can be written in terms of the dressed states:

$$|\Psi(0)\rangle = |\alpha\rangle \otimes |g\rangle = e^{-|\alpha|^2/2}\sum_n \frac{\alpha^n}{\sqrt{n!}}|n;g\rangle = e^{-|\alpha|^2/2}\sum_n \frac{\alpha^n}{\sqrt{n!}}\left(\sin\frac{\theta_n}{2}\,|\phi_n^+\rangle + \cos\frac{\theta_n}{2}\,|\phi_n^-\rangle\right)$$

so that the state's time evolution is given by

$$|\Psi(t)\rangle = e^{-|\alpha|^2/2}\sum_n \frac{\alpha^n}{\sqrt{n!}}\left(\sin\frac{\theta_n}{2}\,|\phi_n^+(t)\rangle + \cos\frac{\theta_n}{2}\,|\phi_n^-(t)\rangle\right) \tag{11.6}$$

where we have incorporated the time dependences into the dressed states:

$$|\phi_n^\pm(t)\rangle = e^{-iE_{n\pm}t/\hbar}|\phi_n^\pm\rangle$$

Now the probability $P_e(t)$ of finding the atom in state $|e\rangle$ at time t is given by $\langle e|\rho_a(t)|e\rangle$, where $\rho_a(t) = \mathrm{Tr}_f[\rho(t)] = \mathrm{Tr}_f[|\Psi(t)\rangle\langle\Psi(t)|]$ is the reduced density matrix for atom. Using (11.6), we find

$$P_e(t) = \langle e|\mathrm{Tr}_f[|\Psi(t)\rangle\langle\Psi(t)|]|e\rangle = \langle e|\sum_n\langle n|\Psi(t)\rangle\langle\Psi(t)|n\rangle|e\rangle = \sum_n |\langle n;e|\Psi(t)\rangle|^2 \tag{11.7}$$

With (25.1) and (11.6) we can readily find apart from a phase factor

$$\langle n;e|\Psi(t)\rangle = e^{-|\alpha|^2/2}\frac{\alpha^{n+1}}{\sqrt{(n+1)!}}\sin\theta_{n+1}\sin\frac{\Omega_{n+1}t}{2}$$

which yields

$$P_e(t) = e^{-|\alpha|^2}\sum_{n=1}^{\infty}\frac{|\alpha|^{2n}}{n!}\sin^2\theta_n\sin^2\frac{\Omega_n t}{2} = \sum_{n=1}^{\infty} p_\alpha(n)\sin^2\theta_n\sin^2\frac{\Omega_n t}{2} \tag{11.8}$$

where $p_\alpha(n) = |\langle n|\alpha\rangle|^2 = e^{-|\alpha|^2}|\alpha|^{2n}/n!$ is the probability that the coherent state has exactly n photons. Similarly we can find that the probability for the atom to be in ground state is

$$P_g(t) = \sum_{n=0}^{\infty} p_\alpha(n)\left(1 - \sin^2\theta_n\sin^2\frac{\Omega_n t}{2}\right) = 1 - P_e(t)$$

Now let us examine the population inversion $w(t) = P_e(t) - P_g(t)$. For the sake of simplicity, we consider the on-resonance case $\Delta = 0$. From the results above we have

$$w(t) = -e^{-|\alpha|^2} - \sum_{n=1}^{\infty} p_\alpha(n)\cos R_n t \tag{11.9}$$

Let us examine the sum when the coherent state field is strong ($|\alpha| \gg 1$), containing many photons, and thus is closest to a classical field. First we recall that for the Poisson distribution $p_\alpha(n)$, the average and variation of the photon number is given by

$$\langle n \rangle = |\alpha|^2, \quad \langle (\Delta n) \rangle = \sqrt{\langle n \rangle} = |\alpha|$$

The sum in (11.9) cannot be evaluated analytically, so the calculation has to be done numerically. An example is shown in Fig. 11-2. Near to $t = 0$ the curves show Rabi oscillations of with an average Rabi period given by $\bar{R}T = 2\pi$, where $\bar{R} = 2g\sqrt{\langle n \rangle} = 2g|\alpha|$ is the average Rabi frequency.

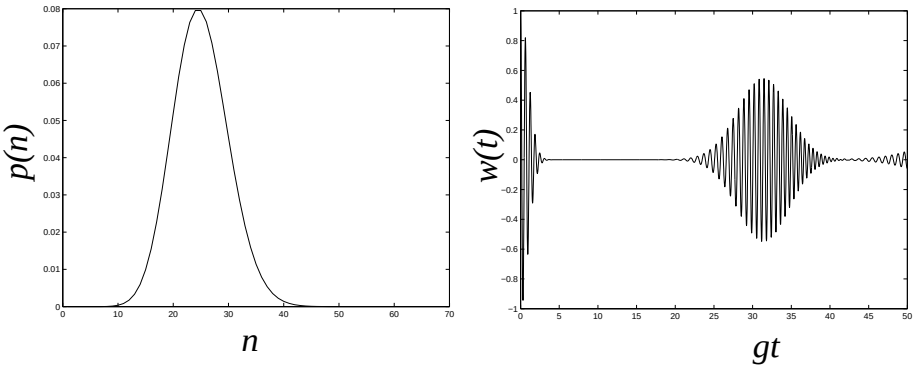

Figure 11-2: Collapse and revival in JC Model. Here the coherent state has $\langle \hat{n} \rangle = 25$.

After some time there is a collapse of the envelope of Rabi oscillations and $w \to 0$. This can be understood as the sum includes contributions from many different Rabi frequencies. When enough time has passed, different oscillatory terms get out-of-phase and cancel with each other. The onset of this "quantum collapse" can be estimated as

$$\delta\bar{R}\tau_{\text{col}} \approx \pi$$

where $\delta\bar{R}$ is the width of the Rabi frequency distribution. The above condition indicates that terms in the sum that are about half the distribution's width away from each other are π out of phase. So we can use the following to estimate $\delta\bar{R}$:

$$\delta\bar{R} = R_{\langle n \rangle + \langle (\Delta n) \rangle/2} - R_{\langle n \rangle - \langle (\Delta n) \rangle/2} = R_{\langle n \rangle + \sqrt{\langle n \rangle}/2} - R_{\langle n \rangle - \sqrt{\langle n \rangle}/2}$$

For $\langle n \rangle \gg 1$, we can expand $R_{\langle n \rangle \pm \sqrt{\langle n \rangle}/2}$ around $\langle n \rangle$ which gives

$$R_{\langle n \rangle \pm \sqrt{\langle n \rangle}/2} = 2g\sqrt{\langle n \rangle \pm \sqrt{\langle n \rangle}/2} = 2g\sqrt{\langle n \rangle}\left(1 \pm \frac{1}{4\sqrt{\langle n \rangle}}\right)$$

Thus we have $\delta\bar{R} = g$ and the collapse time is then

$$\tau_{\text{col}} = \pi/g$$

which is independent of α as long as $|\alpha| \gg 1$.

The truly remarkable feature of the curve is the later restoration to nearly its initial value. This phenomenon is called "revival". The reason for this revival is that the frequencies in the sum are discretized and near the center of the distribution, the frequencies are approximately equally spaced. For large $|\alpha|$, the center will be very broad and will contain many terms. These terms will regain a common phase if every term gets 2π ahead of the phase of the preceding term in the sum, i.e., when $\delta\omega\,\tau_{\rm rev} = 2\pi$, where $\delta\omega = R_{n+1} - R_n = 2g(\sqrt{n+1} - \sqrt{n})$ evaluated near the center of distribution with $n = \langle n\rangle = |\alpha|^2$. Hence we have $\delta\omega = 2g(\sqrt{n+1} - \sqrt{n})|_{n=\langle n\rangle} \approx g/\sqrt{\langle n\rangle}$, and the revival time is given by

$$\tau_{\rm rev} = 2\pi\sqrt{\langle n\rangle}/g$$

i.e., after $\sqrt{\langle n\rangle}$ Rabi periods.

In fact there is an infinite sequence of equally spaced revivals. Revivals are a purely quantized-field phenomenon, arising from the fact that the photon number distribution is not continuous, whereas the collapse is completely classical. These revivals were identified and named and their significance explained in 1980 by Eberly and co-workers after almost 15 years of repeated near-discovery in more and more accurate numerical evaluations of the sum in (11.9).

Quantum collapse and revival was first observed experimentally in 1987 by Rempe and Walter [PRL **58**, 353 (1987)].

Chapter 12

Dressed States: Applications

In describing a dynamical system in quantum optics, we usually start with a set of bare states, which are eigenstates of a time-independent Hamiltonian in the absence of interaction. When the interaction is strong, the bare states are usually not very convenient to use. Under such condition, one usually uses the dressed states introduced in the study of JC model. Dressed states are eigenstates of the total Hamiltonian including interactions. Often times, we find that physics becomes more transparent in the dressed state picture. In this chapter, we will see a few more examples.

12.1 Pump-Probe Spectroscopy of a Two-Level Atom

Consider a two-level atom is coupled to a strong pump field and a weak probe field. By weak we mean the the probe field can be treated as a perturbation of the Pump+Atom system, and the latter can be most conveniently described by the dressed states $|\phi_n^{\pm}\rangle$, where

$$|\phi_n^+\rangle = -\cos\frac{\theta_n}{2}|n-1;e\rangle + \sin\frac{\theta_n}{2}|n;g\rangle \tag{12.1}$$

$$|\phi_n^-\rangle = \sin\frac{\theta_n}{2}|n-1;e\rangle + \cos\frac{\theta_n}{2}|n;g\rangle \tag{12.2}$$

where

$$\cos\frac{\theta_n}{2} = \sqrt{\frac{\Omega_n-\Delta}{2\Omega_n}}, \quad \sin\frac{\theta_n}{2} = \sqrt{\frac{\Omega_n+\Delta}{2\Omega_n}}$$

with $\Omega_n = \sqrt{\Delta^2+R_n^2} = \sqrt{\Delta^2+4ng^2}$. Here n is the number of photons in the pump field and g and Δ are the coupling coefficient and detuning of the pump field, respectively. Suppose the atoms start out in the ground state $|g\rangle$, then the dressed state populations are given by

$$p_n^+ = \sin^2(\theta_n/2), \quad p_n^- = \cos^2(\theta_n/2)$$

In terms of the bare states of the Pump+Atom system, the probe couples the states $|n;g\rangle$ and $|n;e\rangle$. In the dressed basis, this results in a coupling between adjacent manifolds $|\phi_n^{\pm}\rangle$ and $|\phi_{n+1}^{\pm}\rangle$ as shown in Fig. 12-1.

For a strong pump field, that means the pump photon number is centered around certain $n \gg 1$, hence $\theta_{n+1} \approx \theta_n$ and $\Omega_{n+1} \approx \Omega_n$. So the populations in the two manifolds are the same. Note that this is a

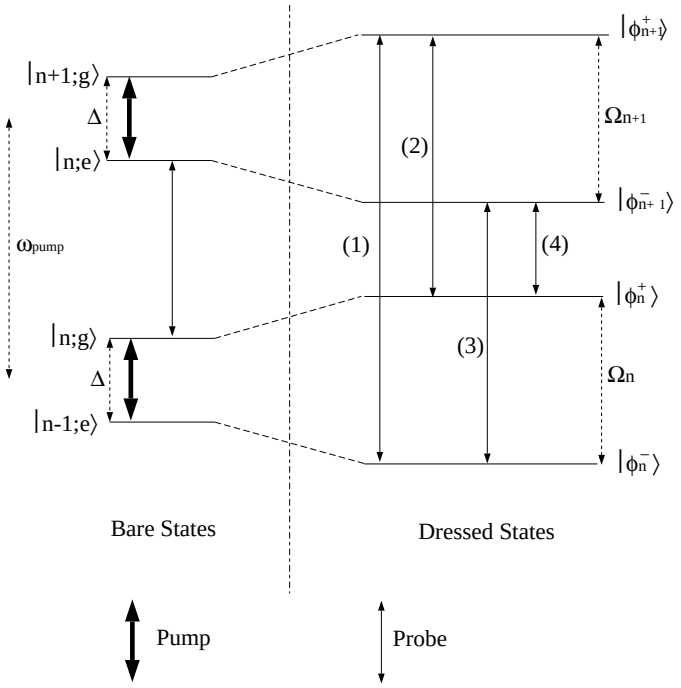

Figure 12-1: Pump-probe spectroscopy in bare state basis and in dressed state basis.

somewhat oversimplified treatment of the steady state population since we have not incorporated decay. Then the four probe induced transitions in the dressed state picture can be described as

1. This transition is resonant when

$$\omega_{\text{probe}} = \omega_{\text{pump}} + \Omega_n$$

   This corresponds to probe absorption if $p_n^- > p_{n+1}^+$ or $\cos^2(\theta_n/2) > \sin^2(\theta_n/2)$ or $\Delta < 0$, while it is probe gain if $\Delta > 0$.

2. This transition is resonant at $\omega_{\text{probe}} = \omega_{\text{pump}}$. There is no probe absorption or gain since $p_n^+ \approx p_{n+1}^+$.

3. Similar to (2), resonant at $\omega_{\text{probe}} = \omega_{\text{pump}}$ and has no probe absorption or gain since $p_n^- \approx p_{n+1}^-$.

4. This transition is resonant at

$$\omega_{\text{probe}} = \omega_{\text{pump}} - \Omega_n$$

   This corresponds to probe absorption if $p_n^+ > p_{n+1}^-$ or $\Delta > 0$, while it is probe gain if $\Delta < 0$.

A typical pump-probe spectroscopy experiment will then show the data as in Fig. 12-2.

12.2 Autler-Townes Doublet

The second example concerns the pump-probe experiment of a three level atom. The upper two levels are coupled by a strong pump field. The frequency of the probe field which couples the lower transition is scanned. One typically gets a double peaked absorption spectrum centered around ω_0 (the lower level

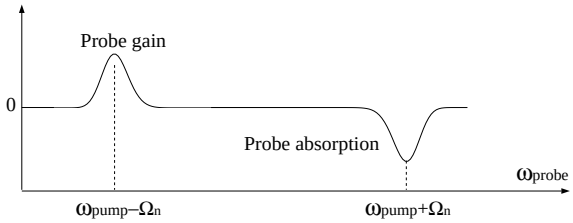

Figure 12-2: Pump-probe spectrum for $\Delta < 0$.

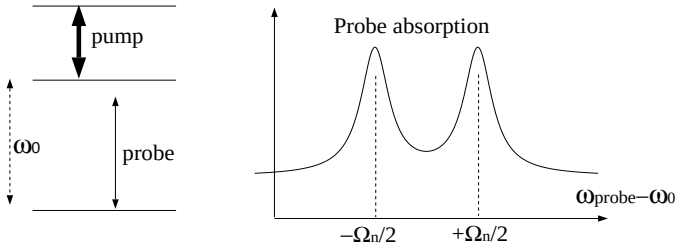

Figure 12-3: Pump-probe spectroscopy showing the Autler-Townes doublet.

transition frequency) and separated by Ω_n (the generalized Rabi frequency of the upper transition). This doublet is called the Autler-Townes doublet. It can be understood easily in the dressed state picture: The two peaks correspond to the excitation of one or the other of the two dressed states involving the upper two levels and the pump field.

Again this description provides us a qualitative picture. The exact form of the spectrum requires the inclusion of spontaneous emission.

12.3 Mollow Triplet in Resonance Fluorescence

Resonance fluorescence refers to the spontaneous emission spectrum of an atom driven by a pump field. For weak pump field, the emission spectrum has a narrow single peak centered at ω_{pump}, which is just the Rayleigh scattering. For strong pump field, however, the spectrum splits into three peaks, one dominant central peak centered at ω_{pump} and two side peaks symmetrically located at $\omega_{\text{pump}} \pm \Omega_n$. This again can be qualitatively understood using the dressed state picture.

In the dressed state picture, the spontaneous emission occurs between adjacent manifolds as shown in Fig. 12-1. These couplings are resonant at frequencies ω_{pump} and $\omega_{\text{pump}} \pm \Omega_n$. Since two such transitions are resonant at the same frequency ω_{pump}, this becomes the most dominant peak in the spectrum. A spectrum as shown in Fig. 12-4 would result. Mollow in 1969 first gave a detailed theoretical calculation of the spectrum, hence the name "Mollow triplet".

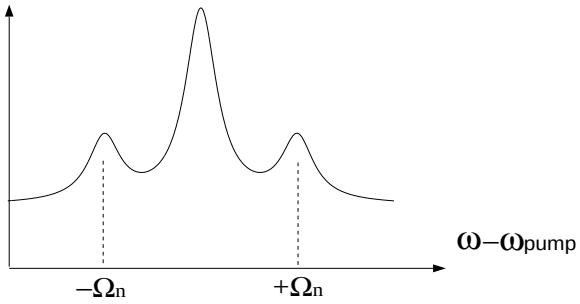

Figure 12-4: Mollow triplet spectrum of resonance fluorescence.

12.4 Dipole force for a Two-Level Atom

In laser cooling and trapping we know that a blue detuned laser repels atoms from the high intensity region due to a repulsive dipole force, and vice versa for a red-detuned laser. This feature of the dipole force can be understood qualitatively in the dressed state picture. Consider a Gaussian laser beam. Due to the spatial dependence of the laser intensity, the energy of the dressed states becomes spatial dependent as shown in Fig. 12-5 for a blue-detuned laser.

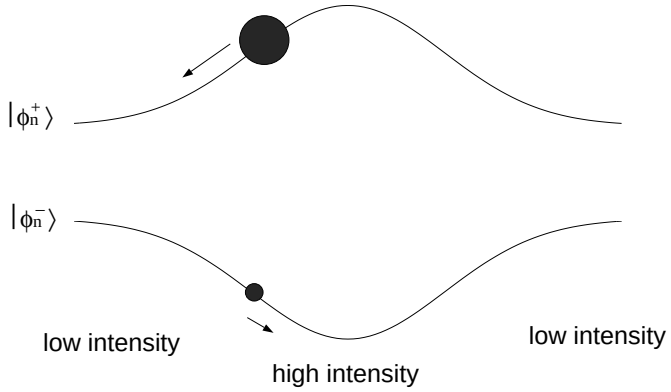

Figure 12-5: Dressed states in a blue detuned Gaussian beam. The circles represents dressed state populations. The size of the circle indicates the relative magnitude of the population.

When the laser is blue-detuned, the upper dressed state $|\phi_n^+\rangle$ contains more ground atomic state $|g\rangle$ which always has a larger population than the excited state $|e\rangle$. As a result, at blue detuning, the upper dressed state is more populated. If we denote the dressed state population by $\Pi_n^{\pm}$, then we have $\Pi_n^+ > \Pi_n^-$.

When the atom moves, the curvature of the dressed state energy gives rise to a force — the dipole force — that is given by

$$\mathbf{F}_{\pm} = -\nabla E_n^{\pm} = \mp\frac{\hbar}{2}\nabla\Omega_n$$

The two dressed states result in two forces with the same magnitude but opposite sign. The force from the upper dressed state repels the atom from the high intensity region; while that from the lower state attracts

the atom to the high intensity region. The net force is the superposition of these forces weighted by their respective populations

$$\mathbf{F} = \Pi_n^+ \mathbf{F}_+ + \Pi_n^- \mathbf{F}_-$$

Since $\Pi_n^+ > \Pi_n^-$, the net force repels the atom from the laser beam.

For red detuning, the opposite is true, the dipole force attracts the atom to the laser beam.

Chapter 13

Spontaneous Emission: Weisskopf-Wigner Theory

It is well known that an atom in an excited state is not in a stationary state — it will eventually decay to the ground state by spontaneously emitting a photon. The nature of this evolution is due to the coupling of the atom to the electromagnetic vacuum field. The idea of spontaneous emission goes back to Einstein when he studied Planck's blackbody spectrum using the principle of detailed balance. The rate of spontaneous emission is still known as the "Einstein A coefficient". Victor Weisskopt presented a method for analyzing this interesting problem in his thesis work, together with his advisor Eugene Wigner. We will follow their treatment here.

Consider a two-level atom. Initially the atom is prepared in its excited state $|e\rangle$ and the field is in vacuum state $|\{0\}\rangle$. We use

$$|\psi(0)\rangle = |e, \{0\}\rangle$$

to denote this initial state. Since this is not a stable state, the atom will decay to the ground state $|g\rangle$ and give off a photon in mode $(\mathbf{k}, s)$. The state of the system after the decay is then $|g, 1_{\mathbf{k}s}\rangle$. These state vectors form a complete set for expanding the time-dependent state of the system:

$$|\psi(t)\rangle = a(t)e^{-i\omega_0 t}|e, \{0\}\rangle + \sum_{\mathbf{k},s} b_{\mathbf{k}s}(t)e^{-i\omega_k t}|g, 1_{\mathbf{k}s}\rangle$$

where ω_0 is the atomic transition frequency and $\omega_k = ck$ is the frequency of the photon.

The total Hamiltonian under the rotating wave approximation is $H = H_A + H_F + H_{\rm int}$ with

$$\begin{aligned} H_A &= \hbar\omega_0\hat{\sigma}_{ee} \\ H_F &= \sum_{\mathbf{k},s} \hbar\omega_k \hat{n}_{\mathbf{k}s} \\ H_{\rm int} &= -\hat{\mathbf{d}}\cdot\hat{\mathbf{E}} = -\sum_{\mathbf{k},s} \hbar g_{\mathbf{k}s}\hat{\sigma}_{eg}\hat{a}_{\mathbf{k}s} + h.c. \end{aligned}$$

where the atom-field coupling coefficient is

$$g_{\mathbf{k}s} = i\sqrt{\frac{\omega_k}{2\hbar\varepsilon_0 V}}\,(\mathbf{d}\cdot\boldsymbol{\epsilon}_{\mathbf{k}s})$$

The Schrödinger equation reads

$$H|\psi(t)\rangle = i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = i\hbar\left(\dot{a} - i\omega_0 a\right)e^{-i\omega_0 t}|e,\{0\}\rangle + i\hbar\sum_{\mathbf{k},s}\left(\dot{b}_{\mathbf{k}s} - i\omega_k b_{\mathbf{k}s}\right)e^{-i\omega_k t}|g,1_{\mathbf{k}s}\rangle$$

By multiply through this equation by $\langle e,\{0\}|$ and $\langle g,1_{\mathbf{k}s}|$, respectively, we obtain

$$\dot{a}(t) = i\sum_{\mathbf{k},s} g_{\mathbf{k}s}e^{-i(\omega_k-\omega_0)t}\,b_{\mathbf{k}s}(t) \tag{13.1}$$

$$\dot{b}_{\mathbf{k}s}(t) = ig^*_{\mathbf{k}s}e^{i(\omega_k-\omega_0)t}\,a(t) \tag{13.2}$$

To solve these equations, we first formally integrate (14.1) as

$$b_{\mathbf{k}s}(t) = ig^*_{\mathbf{k}s}\int_0^t dt'\,e^{i(\omega_k-\omega_0)t'}\,a(t')$$

and put this back into (15.8), we have

$$\dot{a}(t) = -\sum_{\mathbf{k},s}|g_{\mathbf{k}s}|^2\int_0^t dt'\,e^{-i(\omega_k-\omega_0)(t-t')}\,a(t') \tag{13.3}$$

First let us concentrate on $\sum_{\mathbf{k},s}|g_{\mathbf{k}s}|^2$. In the continuum limit (i.e., when the quantization volume $V\to\infty$), we have

$$\sum_{\mathbf{k},s}\to\sum_{s=1}^{2}\int d^3k\mathcal{D}(k)$$

where $\mathcal{D}(k)$ is the density of states in $\mathbf{k}$-space. Since $\mathbf{k} = (2\pi n_1/L, 2\pi n_2/L, 2\pi n_3/L)$, there is one state in volume $(2\pi/L)^3 = (2\pi)^3/V$, hence the density of states is $\mathcal{D}(k) = V/(2\pi)^3$. Then using the spherical coordinates (k,θ,φ), we have

$$\sum_{\mathbf{k},s}\to\sum_{s=1}^{2}\frac{V}{(2\pi)^3}\int_0^\infty k^2dk\int_0^\pi\sin\theta d\theta\int_0^{2\pi}d\varphi$$

Thus

$$\sum_{\mathbf{k},s}|g_{\mathbf{k}s}|^2 = \sum_{\mathbf{k},s}\frac{\omega_k}{2\varepsilon_0 V\hbar}(\mathbf{d}\cdot\boldsymbol{\epsilon}_{\mathbf{k}s})^2 = \int_0^\infty dk\,k^2\frac{\omega_k}{2(2\pi)^3\varepsilon_0\hbar}\left[\sum_{s=1}^{2}\int_0^\pi\sin\theta d\theta\int_0^{2\pi}d\varphi\,(\mathbf{d}\cdot\boldsymbol{\epsilon}_{\mathbf{k}s})^2\right]$$

Here we assume that $\mathbf{d}$ is real, but the final result is more general and works also for complex $\mathbf{d}$. First let us evaluate the quantity inside the square bracket using a simple trick:

$$\sum_{s=1}^{2}\int_0^\pi\sin\theta d\theta\int_0^{2\pi}d\varphi\,(\mathbf{d}\cdot\boldsymbol{\epsilon}_{\mathbf{k}s})^2 = \int_0^\pi\sin\theta d\theta\int_0^{2\pi}d\varphi\,\left[(\mathbf{d}\cdot\boldsymbol{\epsilon}_{\mathbf{k}1})^2 + (\mathbf{d}\cdot\boldsymbol{\epsilon}_{\mathbf{k}2})^2\right] \tag{13.4}$$

Since the triplet $(\boldsymbol{\epsilon}_{\mathbf{k}1},\boldsymbol{\epsilon}_{\mathbf{k}2},\boldsymbol{\kappa})$ with $\boldsymbol{\kappa} = \mathbf{k}/k$ forms an orthogonal set of nit vectors that we can use to expand any vector, so in particular

$$\mathbf{d} = (\mathbf{d}\cdot\boldsymbol{\epsilon}_{\mathbf{k}1})\boldsymbol{\epsilon}_{\mathbf{k}1} + (\mathbf{d}\cdot\boldsymbol{\epsilon}_{\mathbf{k}2})\boldsymbol{\epsilon}_{\mathbf{k}2} + (\mathbf{d}\cdot\boldsymbol{\kappa})\boldsymbol{\kappa}$$

and thus

$$|\mathbf{d}|^2 = (\mathbf{d}\cdot\boldsymbol{\epsilon}_{\mathbf{k}1})^2 + (\mathbf{d}\cdot\boldsymbol{\epsilon}_{\mathbf{k}2})^2 + (\mathbf{d}\cdot\boldsymbol{\kappa})^2$$

or

$$(\mathbf{d}\cdot\boldsymbol{\epsilon}_{\mathbf{k}1})^2 + (\mathbf{d}\cdot\boldsymbol{\epsilon}_{\mathbf{k}2})^2 = |\mathbf{d}|^2 - (\mathbf{d}\cdot\boldsymbol{\kappa})^2$$

We can choose the spherical axis in our integral in any direction that we like, so that we may as well choose it to lie along the direction parallel to $\mathbf{d}$. So we have finally

$$(\mathbf{d}\cdot\boldsymbol{\epsilon}_{\mathbf{k}1})^2+(\mathbf{d}\cdot\boldsymbol{\epsilon}_{\mathbf{k}2})^2=|\mathbf{d}|^2\left(1-\cos^2\theta\right)=|\mathbf{d}|^2\sin^2\theta$$

Now Eq. (13.4) can be easily evalued to give

$$\sum_{s=1}^{2}\int_0^{\pi}\sin\theta d\theta\int_0^{2\pi}d\varphi\,(\mathbf{d}\cdot\boldsymbol{\epsilon}_{\mathbf{k}s})^2=\frac{8\pi}{3}|\mathbf{d}|^2$$

Therefore

$$\sum_{\mathbf{k},s}|g_{\mathbf{k}s}|^2=\int_0^{\infty}dk\,k^2\frac{\omega_k}{2(2\pi)^3\varepsilon_0\hbar}\frac{8\pi}{3}|\mathbf{d}|^2=\frac{|\mathbf{d}|^2}{6\pi^2\varepsilon_0\hbar c^3}\int_0^{\infty}\omega_k^3\,d\omega_k \tag{13.5}$$

where we have changed the integration over k to over $\omega_k=ck$.

Next let us take a look at the time integral in (13.3):

$$\int_0^t dt'\,e^{-i(\omega_k-\omega_0)(t-t')}\,a(t')$$

The exponential oscillates with frequency $\sim\omega_0$. We assume that the excited state amplitude $a(t)$ varies with a rate $\Gamma\ll\omega_0$. Therefore $a(t)$ changes little in the time interval over which the remaining part of the integrand has non-zero value ($t'\sim t$), and we can replace $a(t')$ in the integrand by $a(t)$ and take it out of the integral. This is called the Weisskopf-Wigner approximation, which can be recognized as a Markov approximation: Dynamics of $a(t)$ depends only on time t and not on $t'<t$, i.e., the system has *no memory of the past*. We will come back to Markov approximation in later lectures.

Now the time integral becomes

$$\int_0^t dt'\,e^{-i(\omega_k-\omega_0)(t-t')}\,a(t')\approx a(t)\int_0^t d\tau\,e^{-i(\omega_k-\omega_0)\tau}$$

with $\tau=t-t'$. Since $a(t)$ varies with a rate $\Gamma\ll\omega_0$, the time of interest $t\gg1/\omega_0$, thus we can take the upper limit of the above integral to ∞, and we have

$$\int_0^{\infty}d\tau\,e^{-i(\omega_k-\omega_0)\tau}=\pi\delta(\omega_k-\omega_0)-i\mathcal{P}\left(\frac{1}{\omega_k-\omega_0}\right)$$

where $\mathcal{P}$ represents the Cauchy principal part.

Because of the i before it, the Cauchy principal part leads to a frequency shift. This is in fact one contribution to the Lamb's shift. This shift diverges and has be dealt with through renormalization. Here we will neglect this part. Put things together into (13.3), we finally have

$$\dot{a}(t)=-\frac{\Gamma}{2}a(t)$$

where

$$\Gamma=\frac{\omega_0^3|\mathbf{d}|^2}{3\pi\varepsilon_0\hbar c^3}$$

The excited state amplitude thus decays exponentially as

$$a(t)=e^{-\Gamma t/2}a(0)$$

Γ is then the population decay rate, also known as the Einstein A coefficient.

The Weisskopf-Wigner theory thus predicts an *irreversible* exponential decay of the excited state population with no revivals, in contrast to the JC model. In the latter, revival occurs due to the interaction with a single mode and the discrete nature of the possible photon numbers. In free-space spontaneous emission, the atom is coupled to a *continuum* of modes. Although under the action of each individual mode the atom would have a finite probability to return to the excited state, the probability amplitudes for such events interfere destructively when summer over all the modes.

Finally we can the lineshape of the emitted light. Using (14.1) we have

$$\dot{b}_{\mathbf{k}s}(t) = ig^*_{\mathbf{k}s} e^{i(\omega_k - \omega_0)t}\, a(t) = ig^*_{\mathbf{k}s} e^{i(\omega_k - \omega_0)t} e^{-\Gamma t/2}$$

which can be integrated to give

$$\lim_{t\to\infty} b_{\mathbf{k}s}(t) = \frac{ig^*_{\mathbf{k}s}}{\Gamma/2 - i(\omega_k - \omega_0)}$$

and the corresponding probability is

$$\lim_{t\to\infty} |b_{\mathbf{k}s}(t)|^2 = \frac{|g_{\mathbf{k}s}|^2}{\Gamma^2/4 + (\omega_k - \omega_0)^2}$$

which is a Lorentzian form centered at ω_0 with FWHM Γ.

The angular dependence of the emitted light can be calculated from $\sum_s |\mathbf{d}\cdot\boldsymbol{\epsilon}_{\mathbf{k}s}|^2$. If the dipole transition has $\Delta m = 0$, then $\mathbf{d}$ can be taken as real and can be chosen to lie along the polar axis (z-axis), as we have done above. Thus we have

$$\lim_{t\to\infty} \sum_s |b_{\mathbf{k}s}(t)|^2 \sim \sin^2\theta$$

which is the familiar linear dipole radiation pattern. If, on the other hand, the transition is $\Delta m = \pm 1$, then $\mathbf{d}$ is complex and can be taken as

$$\mathbf{d} = \frac{|\mathbf{d}|}{\sqrt{2}}(\hat{x} \pm i\hat{y})$$

Using the same trick, we have

$$\sum_s |\mathbf{d}\cdot\boldsymbol{\epsilon}_{\mathbf{k}s}|^2 = |\mathbf{d}|^2 - |\boldsymbol{\kappa}\cdot\mathbf{d}|^2 = |\mathbf{d}|^2\left(1 - \frac{\sin^2\theta}{2}\right) = |\mathbf{d}|^2\frac{1+\cos^2\theta}{2}$$

Chapter 14

Quantum Beats

Quantum interference occur when there are two or more undistinguished possible paths in the evolution of a system that leads to the same final state. Such a situation can occur in the process of free space spontaneous emission, and is known as *quantum beats*.

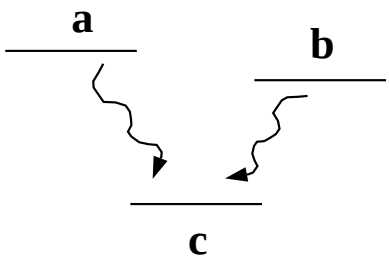

Figure 14-1: Level diagram of a three level atom.

Consider the three level system depicted in the figure. The three levels are labelled as $|a\rangle$, $|b\rangle$ and $|c\rangle$ with allowed transitions between $|a\rangle - |c\rangle$ and $|b\rangle - |c\rangle$. The Hamiltonian is again decomposed into three parts: $H = H_A + H_F + H_{\rm int}$ with

$$\begin{aligned} H_A &= \hbar\omega_a|a\rangle\langle a| + \hbar\omega_b|b\rangle\langle b| \\ H_F &= \sum_{\mathbf{k}} \hbar\omega_k \hat{n}_{\mathbf{k}} \\ H_{\rm int} &= -\sum_{\mathbf{k}} [\hbar g_{\mathbf{k}a}|c\rangle\langle a|\hat{a}_{\mathbf{k}} + \hbar g_{\mathbf{k}b}|c\rangle\langle b|\hat{a}_{\mathbf{k}} + h.c.] \end{aligned}$$

For simplicity, we have neglected the polarization dependence, and

$$g_{\mathbf{k}\alpha} = \frac{d_\alpha \mathcal{E}_k}{\hbar}, \quad (\alpha = a, b)$$

where d_α is the dipole moment for the $|\alpha\rangle - |c\rangle$ transition and $\mathcal{E}_\alpha = \sqrt{\hbar\omega_k/(2\varepsilon_0 V)}$.

Initially the atom is prepared in a superposition of the two upper states while the field is in vacuum state, i.e.,

$$|\psi(0)\rangle = C_a(0)|a, \{0\}\rangle + C_b(0)|b, \{0\}\rangle$$

and we assume that at later time, the state of the system has the form:

$$|\psi(t)\rangle = C_a(t)e^{-i\omega_a t}|a,\{0\}\rangle + C_b(t)e^{-i\omega_b t}|b,\{0\}\rangle + b_k(t)e^{-i\omega_k t}|c,1_{\mathbf{k}}\rangle$$

Following the same procedure as in Weisskopf-Wigner theory, we can derive:

$$\begin{aligned}
\dot{C}_a(t) &= i\sum_{\mathbf{k}} g_{\mathbf{k}a}e^{-i(\omega_k-\omega_a)t}b_{\mathbf{k}}(t) \\
\dot{C}_b(t) &= i\sum_{\mathbf{k}} g_{\mathbf{k}b}e^{-i(\omega_k-\omega_b)t}b_{\mathbf{k}}(t) \\
\dot{b}_{\mathbf{k}}(t) &= ig^*_{\mathbf{k}a}e^{i(\omega_k-\omega_a)t}C_a(t) + ig^*_{\mathbf{k}b}e^{i(\omega_k-\omega_b)t}C_b(t)
\end{aligned}$$

The last equation can be integrated as

$$b_{\mathbf{k}}(t) = ig^*_{\mathbf{k}a}\int_0^t dt'\, e^{i(\omega_k-\omega_a)t'}C_a(t') + ig^*_{\mathbf{k}b}\int_0^t dt'\, e^{i(\omega_k-\omega_b)t'}C_b(t') \tag{14.1}$$

Putting this back into the first equation, we have

$$\dot{C}_a(t) = -\sum_{\mathbf{k}}|g_{\mathbf{k}a}|^2\int_0^t dt'\, e^{-i(\omega_k-\omega_a)(t-t')}C_a(t') - \sum_{\mathbf{k}} g_{\mathbf{k}a}g^*_{\mathbf{k}b}\int_0^t dt'\, e^{-i(\omega_k-\omega_a)t+i(\omega_k-\omega_b)t'}C_b(t')$$

The second at the rhs describes a multiple scattering events: a photon is emitted in the $|a\rangle \to |c\rangle$ transition and then reabsorbed in the $|c\rangle \to |b\rangle$ transition. When $\omega_a \neq \omega_b$, such processes are non-resonant. We thus ignore such processes in the following.

When this term is neglected, the remaining equation is identical to that obtained in the Weisskopf-Wigner treatment. We thus have

$$C_a(t) = C_a(0)e^{-\gamma_a t/2}, \quad C_b(t) = C_b(0)e^{-\gamma_b t/2}$$

When these are put back into (14.1), we have

$$\begin{aligned}
b_{\mathbf{k}}(t) &= ig^*_{\mathbf{k}a}\int_0^t dt'\, e^{i(\omega_k-\omega_a)t'}C_a(0)e^{-\gamma_a t'/2} + ig^*_{\mathbf{k}b}\int_0^t dt'\, e^{i(\omega_k-\omega_b)t'}C_b(0)e^{-\gamma_b t'/2} \\
&= ig^*_{\mathbf{k}a}\frac{e^{i(\omega_k-\omega_a)t-\gamma_a t/2}-1}{i(\omega_k-\omega_a)-\gamma_a/2}C_a(0) + ig^*_{\mathbf{k}b}\frac{e^{i(\omega_k-\omega_b)t-\gamma_b t/2}-1}{i(\omega_k-\omega_b)-\gamma_b/2}C_b(0)
\end{aligned}$$

Now suppose that we have a broadband photodetector located at $\mathbf{R}$. The signal is going to be proportional to the light intensity given by

$$I(t) = \langle\psi(t)|\hat{E}^{(-)}\hat{E}^{(+)}|\psi(t)\rangle = \left|\sum_{\mathbf{k}}\mathcal{E}_k b_{\mathbf{k}}(t)\, e^{i(\mathbf{k}\cdot\mathbf{R}-\omega_k t)}\right|^2$$

Therefore let us evaluate the quantity $S = \sum_{\mathbf{k}}\mathcal{E}_k b_{\mathbf{k}}(t)\, e^{i(\mathbf{k}\cdot\mathbf{R}-\omega_k t)}$. Once again we take the continuum limit to treat the summation over $\mathbf{k}$. Therefore

$$\begin{aligned}
S &= \sum_{\mathbf{k}}\mathcal{E}_k b_{\mathbf{k}}(t)\, e^{i(\mathbf{k}\cdot\mathbf{R}-\omega_k t)} = i\sum_{\mathbf{k}}\mathcal{E}_k g^*_{\mathbf{k}a}\frac{e^{i(\omega_k-\omega_a)t-\gamma_a t/2}-1}{i(\omega_k-\omega_a)-\gamma_a/2}C_a(0)\, e^{i(\mathbf{k}\cdot\mathbf{R}-\omega_k t)} + (a\to b) \\
&= i\frac{V}{(2\pi)^3}C_a(0)\int_0^\infty k^2dk\int_0^\pi \sin\theta d\theta\int_0^{2\pi} d\varphi\frac{d_a\mathcal{E}_k^2}{\hbar}\frac{e^{i(\omega_k-\omega_a)t-\gamma_a t/2}-1}{i(\omega_k-\omega_a)-\gamma_a/2}\, e^{i(\mathbf{k}\cdot\mathbf{R}-\omega_k t)} + (a\to b)
\end{aligned}$$

Now we can choose $\mathbf{R}$ to be along the polar axis, then $e^{i\mathbf{k}\cdot\mathbf{R}} = e^{ikR\cos\theta}$. The angular integral yields

$$\int_0^\pi \sin\theta d\theta\int_0^{2\pi} d\varphi\, e^{i\mathbf{k}\cdot\mathbf{R}} = 2\pi\frac{e^{ikR}-e^{-ikR}}{ikR}$$

Thus

$$S = \frac{C_a(0)d_a}{2(2\pi)^2\varepsilon_0 R}\int_0^\infty dk\,\omega_k k \frac{e^{i(\omega_k-\omega_a)t-\gamma_a t/2}-1}{i(\omega_k-\omega_a)-\gamma_a/2}(e^{ikR}-e^{-ikR})\,e^{-i\omega_k t}+(a\to b)$$

Since the integrand peaks at $\omega_k = \omega_a$, we can put $\omega_k k = \omega_a^2/c$ and take it out of the integral, then the rest integral can be carried out using contour integration

$$\begin{aligned} S &= \frac{C_a(0)\omega_a^2 d_a}{2(2\pi)^2\varepsilon_0 cR}\int_0^\infty dk\,\omega_k k \frac{e^{i(\omega_k-\omega_a)t-\gamma_a t/2}-1}{i(\omega_k-\omega_a)-\gamma_a/2}(e^{ikR}-e^{-ikR})\,e^{-i\omega_k t}+(a\to b) \\ &= C_a(0)\frac{d_a\omega_a^2}{4\pi\varepsilon_0 c^2 R}\,e^{-(\gamma_a/2-i\omega_a)(t-R/c)}\Theta(t-R/c)+(a\to b) \end{aligned}$$

Finally we have for the intensity

$$I(t) = |S|^2 = \frac{\Theta(t-R/c)}{(4\pi\varepsilon_0 c^2 R)^2}\left[|C_a^2(0)d_a\omega_a^2|^2 e^{-\gamma_a(t-R/c)} + C_a(0)C_b^*(0)d_a d_b\omega_a^2\omega_b^2\,e^{-(\gamma_a+\gamma_b)(t-R/c)/2}\,e^{-i\omega_{ab}(t-R/c)}+(a\to b)\right]$$

From this we see that if either $C_a(0)$ or $C_b(0)$ vanishes, we recover the usual exponential decay of the spontaneous transition between the other two levels. But if the system is initially in a coherent superposition of the upper levels, the fluorescence signal has a component oscillating at frequency $\omega_{ab} = \omega_a - \omega_b$. This result is at the origin of a spectroscopic technique used to determine the difference in frequency between two levels. The reason for the existence of this oscillation is due to the quantum interference of the spontaneously emitted photon. When the photon is detected, we don't know whether it originates from the $|a\rangle \to |c\rangle$ transition or the $|b\rangle \to |c\rangle$ transition. This is like the Young's double slit situation, we don't know which slit the photon passed through. Whenever the question "which way" cannot be answered, we get interference!

Chapter 15

Dissipative Processes: System-Reservoir Interaction

In realistic situations, dissipation is always present. The presence of dissipation usually makes an initially pure state become mixed state. Hence in Schrödinger picture, one has to use density operator to describe the system of interest. The equation governing the time evolution of the density operator is called the **Master Equation**.

Dissipation originates from the coupling to the environment (often called the reservoir) which comprises a much larger system or ensemble of states. This coupling is typically weak compared with couplings within the system but may have a rather significant effect on the system, but negligible effect on the reservoir. The effect of the dissipation does not, in general, depend on the precise form of the reservoir nor on the details of the coupling. One such example is the spontaneous emission of an excited atom. The reservoir in this case is just the electromagnetic vacuum.

The total Hamiltonian including both the system and the reservoir reads

$$\mathcal{H} = \mathcal{H}_s + \mathcal{H}_r + \mathcal{H}_I$$

where each component has the following general form

$$\mathcal{H}_s = \hbar\omega_s \hat{s}^\dagger \hat{s}, \quad \mathcal{H}_r = \sum_j \hbar\omega_j \hat{b}_j^\dagger \hat{b}_j, \quad \mathcal{H}_I = \hbar \sum_j \left(g_j \, \hat{s}^\dagger \hat{b}_j + g_j^* \, \hat{b}_j^\dagger \hat{s} \right)$$

where $\hat{s}$ and $\hat{s}^\dagger$ respectively annihilate and create a quantum of energy $\hbar\omega_s$ in the system, and $\hat{b}_j$ and $\hat{b}_j^\dagger$ respectively annihilate and create a quantum of energy $\hbar\omega_j$ in the reservoir.

15.1 Schrödinger Picture Description: Master Equation

In the Schrödinger picture, the goal is to find the evolution of the density operator. We denote ρ_T, ρ and ρ_r as the density operator for the total, system and reservoir, respectively. At the initial time $t = 0$, the system

and the reservoir are not correlated, hence we have $\rho_T(0) = \rho_r(0) \otimes \rho(0)$. This won't be the case for later times.

The total density operator evolves as

$$\dot{\rho}_T = -\frac{i}{\hbar}[\mathcal{H}, \rho_T]$$

Since we want to concentrate on the interaction between the system and the reservoir, it is convenient to change to the interaction picture. Here we take $\mathcal{H}_0 = \mathcal{H}_s + \mathcal{H}_r$, then in the interaction picture, we have

$$\rho_T^{(I)}(t) = e^{i\mathcal{H}_0 t/\hbar}\rho_T(t)e^{-i\mathcal{H}_0 t/\hbar} \tag{15.1}$$

$$V_I(t) = e^{i\mathcal{H}_0 t/\hbar}\mathcal{H}_I e^{-i\mathcal{H}_0 t/\hbar} = \hbar\sum_j g_j\, \hat{s}^\dagger \hat{b}_j e^{-i(\omega_j-\omega_s)t} + h.c. = i\hbar \hat{s}^\dagger \hat{F}(t) + h.c. \tag{15.2}$$

$$\dot{\rho}_T^{(I)}(t) = -\frac{i}{\hbar}[V_I(t), \rho_T^{(I)}(t)] \tag{15.3}$$

where we have defined the *noise operator*

$$\hat{F}(t) = -i\sum_j g_j\, \hat{b}_j e^{-i(\omega_j-\omega_s)t} \tag{15.4}$$

Equation (15.3) can be integrated formally as [we now drop the superscript (I) for notational convenience]:

$$\rho_T(t) = \rho_T(0) - \frac{i}{\hbar}\int_0^t dt'\,[V_I(t'), \rho_T(t')]$$

This can be solve iteratively. For example, let us replace $\rho_T(t')$ at the r.h.s. by $\rho_T(t') = \rho_T(0) - (i/\hbar)\int_0^{t'} dt''\,[V_I(t''), \rho_T(t'')]$ and obtain the second order equation:

$$\rho_T(t) = \rho_T(0) - \frac{i}{\hbar}\int_0^t dt'\,[V_I(t'), \rho_T(0)] - \frac{1}{\hbar^2}\int_0^t dt' \int_0^{t'} [V_I(t'), [V_I(t''), \rho_T(t'')]]$$

This process can go on and on, but it usually suffices to truncate at the second order. The truncation is to replace $\rho_T(t'')$ at the r.h.s. by a decorrelated form

$$\rho_T(t'') \approx \rho_r(t'') \otimes \rho(t'') \approx \rho_r(0) \otimes \rho(t'')$$

where we have used the fact the system-reservoir interaction does not change the reservoir significantly, hence $\rho_r(t'') \approx \rho_r(0)$. This truncation procedure is called the *second-order Born approximation.*

Now we can rewrite the equation of motion for ρ_T as

$$\dot{\rho}_T(t) = -\frac{i}{\hbar}[V_I(t), \rho_r(0) \otimes \rho(0)] - \frac{1}{\hbar^2}\int_0^t dt'\,[V_I(t), [V_I(t'), \rho_r(0) \otimes \rho(t')]] \tag{15.5}$$

What we are really interested in is the dynamics of the system, or $\rho(t)$. To this end, we find the equation of motion for $\rho(t)$ by trace over the reservoir degrees of freedom from the above equation.

Performing this trace on the first term at the r.h.s. of (15.5) we have

$$\mathrm{Tr}_r\{[V_I(t), \rho_r(0) \otimes \rho(0)]\} = \mathrm{Tr}_r\{[V_I(t), \rho_r(0)]\}\rho(0)$$

which results in terms like $\langle \hat{F} \rangle$ and $\langle \hat{F}^\dagger \rangle$. For a reservoir in thermal equilibrium, ρ_r is diagonal under Fock state basis, hence

$$\langle \hat{F} \rangle = 0 = \langle \hat{F}^\dagger \rangle$$

Therefore to second order, the master equation becomes

$$\begin{aligned}\dot{\rho}(t) &= -\frac{1}{\hbar^2}\int_0^t dt'\,\mathrm{Tr}_r\{[V_I(t),[V_I(t'),\rho_T(t')]]\} \\ &= \int_0^t dt'\,\mathrm{Tr}_r\{[\hat{s}\hat{F}(t)-\hat{F}^\dagger(t)\hat{s},[\hat{s}\hat{F}(t')-\hat{F}^\dagger(t')\hat{s},\rho_r(0)\otimes\rho(t')]]\}\end{aligned}$$

Expanding the commutators gives 16 terms. Using $\mathrm{Tr}_r[\hat{F}(t)\rho_r\hat{F}^\dagger(t')]=\mathrm{Tr}_r[\rho_r\hat{F}^\dagger(t')\hat{F}(t)]=\langle\hat{F}^\dagger(t')\hat{F}(t)\rangle$, we have

$$\begin{aligned}\dot{\rho}(t) &= \int_0^t dt'\Big\{[\hat{s}\rho(t')\hat{s}^\dagger-\hat{s}^\dagger\hat{s}\rho(t')]\langle\hat{F}(t)\hat{F}^\dagger(t')\rangle+[\hat{s}\rho(t')\hat{s}^\dagger-\rho(t')\hat{s}^\dagger\hat{s}]\langle\hat{F}(t')\hat{F}^\dagger(t)\rangle \\ &\quad+[\hat{s}^\dagger\rho(t')\hat{s}-\hat{s}\hat{s}^\dagger\rho(t')]\langle\hat{F}^\dagger(t)\hat{F}(t')\rangle+[\hat{s}^\dagger\rho(t')\hat{s}-\rho(t')\hat{s}\hat{s}^\dagger]\langle\hat{F}^\dagger(t')\hat{F}(t)\rangle \\ &\quad-[\hat{s}^\dagger\rho(t')\hat{s}^\dagger-\hat{s}^\dagger\hat{s}^\dagger\rho(t')]\langle\hat{F}(t)\hat{F}(t')\rangle-[\hat{s}^\dagger\rho(t')\hat{s}^\dagger-\rho(t')\hat{s}^\dagger\hat{s}^\dagger]\langle\hat{F}(t')\hat{F}(t)\rangle \\ &\quad-[\hat{s}\rho(t')\hat{s}-\hat{s}\hat{s}\rho(t')]\langle\hat{F}^\dagger(t)\hat{F}^\dagger(t')\rangle-[\hat{s}\rho(t')\hat{s}-\rho(t')\hat{s}\hat{s}]\langle\hat{F}^\dagger(t')\hat{F}^\dagger(t)\rangle\Big\}\end{aligned}$$

Under the Markov approximation, the noise operators are δ-correlated, with

$$\begin{aligned}\langle\hat{F}(t)\hat{F}^\dagger(t')\rangle &= \langle\hat{F}(t')\hat{F}^\dagger(t)\rangle=\frac{\Gamma}{2}(\bar{n}+1)\,\delta(t-t'),\quad \langle\hat{F}^\dagger(t)\hat{F}(t')\rangle=\langle\hat{F}^\dagger(t')\hat{F}(t)\rangle=\frac{\Gamma}{2}\bar{n}\,\delta(t-t') \\ \langle\hat{F}(t)\hat{F}(t')\rangle &= \langle\hat{F}(t')\hat{F}(t)\rangle=\frac{\Gamma}{2}M\,\delta(t-t'),\quad \langle\hat{F}^\dagger(t)\hat{F}^\dagger(t')\rangle=\langle\hat{F}^\dagger(t')\hat{F}^\dagger(t)\rangle=\frac{\Gamma}{2}M^*\,\delta(t-t')\end{aligned}$$

with $\Gamma=2\pi\sum_j|g_j|^2\delta(\omega_j-\omega_s)$, $\bar{n}=\langle\hat{b}^\dagger(\omega_s)\hat{b}(\omega_s)\rangle$ and $M=\langle\hat{b}(\omega_s)\hat{b}(\omega_s)\rangle$. The derivations follow a similar procedure as in the discussion of the WW theory of spontaneous emission. Here we see that Γ, which characterize the decay rate of the system, is also related to the noise correlations which characterize the fluctuations of the reservoir. This property is a consequence of the so-called *fluctuation-dissipation theorem.*

From these relations, the master equation becomes

$$\begin{aligned}\dot{\rho}(t) &= \frac{\Gamma}{2}(\bar{n}+1)[2\hat{s}\rho(t)\hat{s}^\dagger-\hat{s}^\dagger\hat{s}\rho(t)-\rho(t)\hat{s}^\dagger\hat{s}]+\frac{\Gamma}{2}\bar{n}[2\hat{s}^\dagger\rho(t)\hat{s}-\hat{s}\hat{s}^\dagger\rho(t)-\rho(t)\hat{s}\hat{s}^\dagger] \\ &\quad-\frac{\Gamma}{2}M[2\hat{s}^\dagger\rho(t)\hat{s}^\dagger-\hat{s}^\dagger\hat{s}^\dagger\rho(t)-\rho(t)\hat{s}^\dagger\hat{s}^\dagger]-\frac{\Gamma}{2}M^*[2\hat{s}\rho(t)\hat{s}-\hat{s}\hat{s}\rho(t)-\rho(t)\hat{s}\hat{s}]\end{aligned}$$

In general, the positivity of the density operator requires that $|M|^2\le\bar{n}(\bar{n}+1)$. In most cases, the reservoir is in a thermal equilibrium state, then $M=M^*=0$, and $\bar{n}$ is the average number of the thermal quanta. For zero temperature, $\bar{n}=0$ and the master equation is further simplified. A finite M indicates that the reservoir possesses phase sensitive correlations. This arises, for example, in the case of a squeezed reservoir state.

For a thermal reservoir at zero temperature, we have $\bar{n}=0=M$. Going back to the Schrödinger picture, the Master equation can be simplified to

$$\dot{\rho}(t)=-\frac{i}{\hbar}[\mathcal{H}_s,\rho(t)]+\frac{\Gamma}{2}\,[2\hat{s}\rho(t)\hat{s}^\dagger-\hat{s}^\dagger\hat{s}\rho(t)-\rho(t)\hat{s}^\dagger\hat{s}]$$

15.2 Heisenberg Picture Description: Quantum Langevin Equation

To gain more insight into the system-reservoir interaction, we now treat the same problem in the Heisenberg picture. In this treatment, the reservoir operators can be interpreted similarly to Langevin forces in classical

statistical mechanics. These *quantum noise operators* are the source of both fluctuations and irreversible dissipation of energy from the system to the reservoir. Here we will study a specific example of the interaction between a two-level atom and the free space electromagnetic field under the RWA. The Hamiltonian reads

$$\mathcal{H}_s = \hbar\omega_A \hat{\sigma}_{ee}, \quad \mathcal{H}_r = \sum_j \hbar\omega_j \hat{a}_j^\dagger \hat{a}_j, \quad \mathcal{H}_I = \sum_j \hbar g_j \hat{\sigma}_{eg} \hat{a}_j + \sum_j \hbar g_j^* \hat{a}_j^\dagger \hat{\sigma}_{ge}$$

It is again convenient to adopt the interaction picture, in which the Hamiltonian becomes

$$V_I(t) = \sum_j \hbar g_j \hat{\sigma}_{eg} \hat{a}_j e^{-i\Delta_j t} + \sum_j \hbar g_j^* \hat{a}_j^\dagger \hat{\sigma}_{ge} e^{i\Delta_j t}$$

where $\Delta_j = \omega_j - \omega_A$. The Heisenberg equations of motion can be obtained as

$$\dot{\hat{\sigma}}_{ge}(t) = i \sum_j g_j \hat{\sigma}_3(t)\, \hat{a}_j(t)\, e^{-i\Delta_j t} \tag{15.6}$$

$$\dot{\hat{\sigma}}_3(t) = -2i \sum_j g_j \hat{\sigma}_{eg}(t)\, \hat{a}_j(t)\, e^{-i\Delta_j t} + h.c. \tag{15.7}$$

$$\dot{\hat{a}}_j(t) = -i g_j^* \hat{\sigma}_{ge}(t)\, e^{i\Delta_j t} \tag{15.8}$$

where $\hat{\sigma}_3 \equiv \hat{\sigma}_{ee} - \hat{\sigma}_{gg}$.

Equation (15.8) can be formally solved as

$$\hat{a}_j(t) = \hat{a}(0) - i g_j^* \int_0^t dt'\, \hat{\sigma}_{ge}(t')\, e^{i\Delta_j t'}$$

Put this into (15.6), we have

$$\dot{\hat{\sigma}}_{ge}(t) = i \sum_j g_j \hat{\sigma}_3(t)\, \hat{a}_j(0)\, e^{-i\Delta_j t} + \sum_j |g_j|^2 \hat{\sigma}_3(t) \int_0^t dt'\, \hat{\sigma}_{ge}(t')\, e^{i\Delta_j (t'-t)} \tag{15.9}$$

The first term at the r.h.s. can be written as $-\hat{\sigma}_3 \hat{F}(t)$ where $\hat{F}(t) = -i \sum_j g_j a_j(0) e^{-i\Delta_j t}$ is just the noise operator, also known as the Langevin operator, as defined in Eq. (16.5). The second term at the r.h.s. can be treated under the Markov approximation. Assuming that $\sum_j |g_j|^2\, e^{i\Delta_j (t'-t)}$ is sharply peaked at $t' - t = 0$, then we can replace $\hat{\sigma}_{ge}(t')$ by $\hat{\sigma}_{ge}(t)$ and take it out of the integral over t'. Using

$$\int_0^t dt'\, e^{i\Delta_j (t'-t)} \approx \int_0^\infty dt'\, e^{i\Delta_j (t'-t)} = \pi\delta(\Delta_j)$$

the second term becomes $-\Gamma\hat{\sigma}_{ge}(t)/2$, where Γ is the same decay rate we are now familiar with and we have used that $\hat{\sigma}_3(t)\hat{\sigma}_{ge}(t) = -\hat{\sigma}_{ge}(t)$. In conclusion, we have

$$\dot{\hat{\sigma}}_{ge}(t) = -\frac{\Gamma}{2}\, \hat{\sigma}_{ge}(t) - \hat{\sigma}_3(t)\hat{F}(t) \tag{15.10}$$

Similarly we can have

$$\dot{\hat{\sigma}}_{eg}(t) = -\frac{\Gamma}{2}\, \hat{\sigma}_{eg}(t) - \hat{F}^\dagger(t)\hat{\sigma}_3(t) \tag{15.11}$$

$$\dot{\hat{\sigma}}_3(t) = -\Gamma[\hat{\sigma}_3(t) + 1] + 2\hat{\sigma}_{eg}(t)\hat{F}(t) + 2\hat{F}^\dagger(t)\hat{\sigma}_{ge}(t) \tag{15.12}$$

Eqs. (15.10) (15.11) and (18.8) are called the *quantum Langevin equations* because they contain terms proportional to the Langevin noise operators.

These equations can be formally integrated. Put two such formal integration solution to the third equation, take the expectation value and perform a system-noise decorrelation such as

$$\langle\hat{\sigma}_3(t')\hat{F}^\dagger(t')\hat{F}(t)\rangle = \langle\hat{\sigma_3}(t')\rangle\langle\hat{F}^\dagger(t')\hat{F}(t)\rangle$$

we find that we are again in the situation to calculate the correlations of the noise operators which we know how to do under the Markov approximation. Finally we have

$$\dot{\sigma}_{ge}(t) = -\frac{\Gamma}{2}(2\bar{n}+1)\sigma_{ge}(t)\rangle - \Gamma M\sigma_{eg}(t) \tag{15.13}$$

$$\dot{\sigma}_{eg}(t) = -\frac{\Gamma}{2}(2\bar{n}+1)\sigma_{eg}(t) - \Gamma M^*\sigma_{ge}(t) \tag{15.14}$$

$$\dot{\sigma}_3(t) = -\Gamma[(2\bar{n}+1)\sigma_3(t)+1] \tag{15.15}$$

where the quantity without the hat is the expectation value of the corresponding operator.

Using the quantum Langevin equation, it is very convenient to calculate the two-time correlation function. For example, let us evaluate $\langle\hat{\sigma}_{ge}(t)\hat{\sigma}_{eg}(t')\rangle$ and for convenience, assuming $t > t'$. Multiply both sides of (15.10) by $\hat{\sigma}_{eg}(t')$ and take the expectation value, we have

$$\frac{d}{dt}\langle\hat{\sigma}_{ge}(t)\hat{\sigma}_{eg}(t')\rangle = -\frac{\Gamma}{2}\langle\hat{\sigma}_{ge}(t)\hat{\sigma}_{eg}(t')\rangle - \langle\hat{\sigma}_3(t)\hat{F}(t)\hat{\sigma}_{eg}(t')\rangle$$

Let us evaluate the second at the r.h.s. using the identity

$$\hat{\sigma}_3(t) = \hat{\sigma}_3(t-\delta t) + \int_{t-\delta t}^{t} d\tau\,\dot{\hat{\sigma}}_3(\tau)$$

we then have

$$\langle\hat{\sigma}_3(t)\hat{F}(t)\hat{\sigma}_{eg}(t')\rangle = \langle\hat{\sigma}_3(t-\delta t)\hat{F}(t)\hat{\sigma}_{eg}(t')\rangle + \int_{t-\delta t}^{t} d\tau\,\langle\dot{\hat{\sigma}}_3(\tau)\hat{F}(t)\hat{\sigma}_{eg}(t')\rangle$$

The first term at the r.h.s. vanish since the atomic operators at earlier times cannot depend on $\hat{F}(t)$. Using (18.8) we can evaluate the second term at the r.h.s. At this stage we need to decorrelate the atomic and the noise operators. Again, we need to evaluate the noise correlations. After a straightforward procedure, we have

$$\langle\hat{\sigma}_3(t)\hat{F}(t)\hat{\sigma}_{eg}(t')\rangle = \Gamma\bar{n}\langle\hat{\sigma}_{ge}(t)\hat{\sigma}_{eg}(t')\rangle + \Gamma M\langle\hat{\sigma}_{eg}(t)\hat{\sigma}_{eg}(t')\rangle$$

Therefore we finally have

$$\frac{d}{dt}\langle\hat{\sigma}_{ge}(t)\hat{\sigma}_{eg}(t')\rangle = -\frac{\Gamma}{2}(2\bar{n}+1)\langle\hat{\sigma}_{ge}(t)\hat{\sigma}_{eg}(t')\rangle - \Gamma M\langle\hat{\sigma}_{eg}(t)\hat{\sigma}_{eg}(t')\rangle$$

Compare this with (15.13) we find that the two-time correlation function $\langle\hat{\sigma}_{ge}(t)\hat{\sigma}_{eg}(t')\rangle$ satisfy exactly the same equation of motion for the single-time expectation value $\sigma_{ge}(t)$. This is called the *quantum regression theorem*.

Chapter 16

Atomic Relaxation

We have seen in the WW theory of spontaneous decay how a coupling between an two-level atom and the vacuum field leads to relaxation. Here we want to extend this to a more generalized situation.

Consider an atom with multiple levels 1,2,3, ... Level k lies above Level m if $k > m$. We will use a density matrix approach to study in more detail the system-reservoir interaction.

16.1 Equations of Motion

Denote the density operators for the system, reservoir and the total as ρ, ρ_R and ρ_T, respectively. At time $t = 0$, there is no entanglement between the system and the reservoir, hence

$$\rho_T(0) = \rho(0) \otimes \rho_R(0)$$

Let $\hat{B}_\lambda$ and $\hat{B}_\lambda^\dagger$ be some generic reservoir mode operators that satisfy the commutation relation

$$[\hat{B}_\lambda, \hat{B}_\mu^\dagger] = \delta_{\lambda\mu}$$

For the particular example of the familiar $\mathbf{d} \cdot \mathbf{E}$ dipole interaction between an atom and the empty modes of the vacuum, $\hat{B}_\lambda$ would corresponds to $\hat{a}_{\mathbf{k}s}$. When expressed in terms of the $\hat{\sigma}$ operators for the atom, the Hamiltonian describing the atom-reservoir interaction reads

$$H_{\text{int}} = -\hbar \sum_\lambda \sum_{k>m} g(\lambda; k, m) \left(\hat{B}_\lambda^\dagger \hat{\sigma}_{mk} + \hat{\sigma}_{km} \hat{B}_\lambda \right) \tag{16.1}$$

where the coupling parameter $g(\lambda; k, m)$ is taken to be real for simplicity. The symbol $k > m$ is intended to indicate that both labels k and m are summed over all of the atomic states, but with the $k > m$ restriction. This has the effect of directly eliminating from the Hamiltonian any possibility of "counter-rotating" terms (Level k lies above Level m if $k > m$).

The bare Hamiltonians for the atom and the reservoir are given by

$$H_A = \sum_k E_k \hat{\sigma}_{kk}, \quad H_R = \sum_\lambda \hbar\omega_\lambda \hat{B}_\lambda^\dagger \hat{B}_\lambda$$

The dynamical equations for $\hat{\sigma}$'s and $\hat{B}$ operators are easily found as

$$i\dot{\hat{\sigma}}_{lm} = \omega_{ml}\hat{\sigma}_{lm} - \sum_{s>c}\sum_{\lambda} g(\lambda;s,c)(\hat{\sigma}_{lc}\delta_{ms} - \hat{\sigma}_{sm}\delta_{cl})\hat{B}_\lambda + h.c. - \sum_{s>c}\sum_{\lambda} g(\lambda;s,c)\hat{B}^\dagger_\lambda(\hat{\sigma}_{ls}\delta_{mc} - \hat{\sigma}_{cm}\delta_{sl}) \quad (16.2)$$

$$i\dot{\hat{B}}_\lambda = \omega_\lambda \hat{B}_\lambda - \sum_{p>a} g(\lambda;p,a)\hat{\sigma}_{ap} \quad (16.3)$$

where $\hbar\omega_{ml} = E_m - E_l$.

16.2 Preliminary Solutions

From (16.3), the formal solution for $\hat{B}_\lambda(t)$ can be found as

$$\hat{B}_\lambda(t) = \hat{B}_\lambda(0)e^{-i\omega_\lambda t} + i\sum_{p>a} g(\lambda;p,a)\int_0^t dt'\, e^{-i\omega_\lambda(t-t')}\,\hat{\sigma}_{ap}(t')$$

which can be decomposed into two parts $\hat{B}_\lambda(t) = \hat{B}^{\rm self}_\lambda(t) + \hat{B}^{\rm atom}_\lambda(t)$ with

$$\hat{B}^{\rm self}_\lambda(t) = \hat{B}_\lambda(0)e^{-i\omega_\lambda t}$$
$$\hat{B}^{\rm atom}_\lambda(t) = i\sum_{p>a} g(\lambda;p,a)\int_0^t dt'\, e^{-i\omega_\lambda(t-t')}\,\hat{\sigma}_{ap}(t')$$

When we substitute this formal solution into (18.5), we will encounter two terms, namely

$$\sum_\lambda g(\lambda;s,c)\left(\hat{B}^{\rm self}_\lambda + \hat{B}^{\rm atom}_\lambda\right) \quad (16.4)$$

These will have an important relation to each other. But let us first focus on the second term which can rewrite as

$$\sum_\lambda g(\lambda;s,c)\hat{B}^{\rm atom}_\lambda = i\sum_{p>a}\int_0^t dt'\, G_{scpa}(t-t')\hat{\sigma}_{ap}(t')$$

where we have defined

$$G_{scpa}(\tau) \equiv \sum_\lambda g(\lambda;s,c)g(\lambda;p,a)e^{-i\omega_\lambda\tau}$$

which is called the *reservoir response function*. Since the reservoir modes are assumed to be dense, the summation over λ should be understood as an integral over reservoir frequency ω_λ. Hence $G(\tau)$ can be understood as something like the Fourier transform of the square of the atom-reservoir coupling coefficient g.

In addition, the reservoir coupling is assumed to be broadband and without discrete resonances, the functional dependence of g is nearly independent of ω_λ, smooth and almost featureless over a wide band of reservoir frequencies. Thus we can anticipate that the quasi-Fourier transform $G(\tau)$ is almost a δ-function, e.g., non-zero only in an extremely narrow neighborhood of $\tau = 0$. The width of this non-zero region, τ_R, is called the *reservoir coherence time*. When τ_R is much smaller than any time scale of interest, we can consider τ_R to be zero, which is equivalently to saying that the reservoir response is practically instantaneous. This dependence on the present time, to the effective exclusion of earlier times, is just the *Markov approximation*.

The contribution of the first term in (16.4) can be abbreviated as

$$\hat{n}_{sc}(t) = \sum_\lambda g(\lambda;s,c)\hat{B}^{\rm self}_\lambda = \sum_\lambda g(\lambda;s,c)\hat{B}_\lambda(0)\,e^{-i\omega_\lambda t} \quad (16.5)$$

and will be called a reservoir *noise operator*. It contains information about the character of the reservoir.

Putting these back into (18.5), we have

$$\begin{aligned} i\dot{\hat{\sigma}}_{lm} &= \omega_{ml}\hat{\sigma}_{lm} - \left[\sum_{s>c}(\hat{\sigma}_{lc}\delta_{ms} - \hat{\sigma}_{sm}\delta_{cl})\left(\hat{n}_{sc}(t) + i\sum_{p>a}\int_0^t dt'\, G_{scpa}(t-t')\hat{\sigma}_{ap}(t')\right)\right. \\ &\quad \left. + \sum_{s>c}\left(\hat{n}^\dagger_{sc}(t) + i\sum_{p>a}\int_0^t dt'\, G^*_{scpa}(t-t')\hat{\sigma}_{pa}(t')\right)(\hat{\sigma}_{ls}\delta_{mc} - \hat{\sigma}_{cm}\delta_{sl})\right] \end{aligned} \tag{16.6}$$

Obviously, the $\hat{\sigma}$ operators evolve on two time scales, a fast scale associated with free evolution and a much slower scale associated with the weak coupling to the reservoir. The free evolution is given by

$$\hat{\sigma}_{ap}(t) = \hat{\sigma}_{ap}(0)\, e^{-i\omega_{pa}t}$$

where the entire evolution is in the phase, with no change occurring in the amplitude. In (16.6), the presence of $G(t-t')$ in the time integral means that the integrand is zero except in a short time interval, of length τ_R, very near to $t' = t$. During this short time, the amplitude can be assumed not to change, but the rapid phase change of $\hat{\sigma}_{ap}$ must be accounted for as

$$\hat{\sigma}_{ap}(t') \approx \hat{\sigma}_{ap}(t)\, e^{-i\omega_{pa}(t-t')}$$

Therefore we have

$$\begin{aligned} i\dot{\hat{\sigma}}_{lm} &= \omega_{ml}\hat{\sigma}_{lm} - \left[\sum_{s>c}(\hat{\sigma}_{lc}\delta_{ms} - \hat{\sigma}_{sm}\delta_{cl})\left(\hat{n}_{sc}(t) + i\sum_{p>a} g_{scpa}(t)\hat{\sigma}_{ap}(t)\right)\right. \\ &\quad \left. + \sum_{s>c}\left(\hat{n}^\dagger_{sc}(t) + i\sum_{p>a} g^*_{scpa}(t)\hat{\sigma}_{pa}(t)\right)(\hat{\sigma}_{ls}\delta_{mc} - \hat{\sigma}_{cm}\delta_{sl})\right] \end{aligned} \tag{16.7}$$

where

$$g_{scpa}(t) = \int_0^t dt'\, G_{scpa}(t-t')\, e^{-i\omega_{pa}(t-t')} = \sum_\lambda g(\lambda;s,c)g(\lambda;p,a)\int_0^t dt'\; e^{-i(\omega_\lambda - \omega_{pa})(t-t')}$$

The time integral, in the large t limit, yields

$$\int_0^t dt'\; e^{-i(\omega_\lambda-\omega_{pa})(t-t')} \to \pi\delta(\omega_\lambda - \omega_{pa}) - i\mathcal{P}\left(\frac{1}{\omega_\lambda - \omega_{pa}}\right)$$

and subsequently

$$g_{scpa}(t) \to \Gamma_{scpa} - i\delta_{scpa}$$

where

$$\Gamma_{scpa} = \pi\sum_\lambda g(\lambda;s,c)g(\lambda;p,a)\delta(\omega_\lambda - \omega_{pa}) \tag{16.8}$$

$$\delta_{scpa} = \sum_\lambda g(\lambda;s,c)g(\lambda;p,a)\mathcal{P}\left(\frac{1}{\omega_\lambda - \omega_{pa}}\right) \tag{16.9}$$

play the roles of damping rates and frequency shifts, respectively. The RWA theory is not adequate to give the correct numerical value of the frequency shift. We will simply assume that the correct shift has been included in the bare frequencies ω_{ml} and ignore the δ_{scpa} terms from now on.

If we take expectation values on both sides of (16.7), The contribution from the noise operators terms vanish. This is because the noise operators are related to $\hat{B}(0)$ and $\hat{B}^\dagger(0)$ only. These are not correlated to the atomic operators. And since the reservoir is assumed to be in a thermal equilibrium state, its density matrix is diagonal in the reservoir energy representation, hence $\langle\hat{B}(0)\rangle = \langle\hat{B}^\dagger(0)\rangle = 0$. Now we can neglect the noise terms $\hat{n}_{sc}$ in (16.7). The equation for $\hat{\sigma}$ becomes

$$\begin{aligned} i\dot{\hat{\sigma}}_{lm} &= \omega_{ml}\hat{\sigma}_{lm} - i\sum_{s>c}\sum_{p>a}\Gamma_{scpa}\left(\hat{\sigma}_{lp}\delta_{ms}\delta_{ca} - \hat{\sigma}_{sp}\delta_{cl}\delta_{ma}\right) + i\sum_{s>c}\sum_{p>a}\Gamma_{scpa}\left(\hat{\sigma}_{ps}\delta_{al}\delta_{mc} - \hat{\sigma}_{pm}\delta_{ac}\delta_{sl}\right) \\ &= \omega_{ml}\hat{\sigma}_{lm} - i\sum_{s>c}\sum_{p>a}\Gamma_{scpa}\left(\hat{\sigma}_{lp}\delta_{ms}\delta_{ca} - \hat{\sigma}_{sp}\delta_{cl}\delta_{ma} - \hat{\sigma}_{ps}\delta_{al}\delta_{mc} + \hat{\sigma}_{pm}\delta_{ac}\delta_{sl}\right) \end{aligned} \tag{16.10}$$

where we have used $\hat{\sigma}_{lc}\hat{\sigma}_{ap} = \hat{\sigma}_{lp}\delta_{ca}$.

16.3 Diagonal and Off-Diagonal Relaxation

Eq. (16.10) can be simplified by organizing the multiple summations into two kinds of terms — those that oscillate at the same frequency as $\hat{\sigma}_{lm}$ and those that don't. Reservoir theory simply discards those of the second kind, on the grounds that they rapidly get out of phase and make negligible long-term contributions. With this in mind, we will consider the off-diagonal or "coherence" terms ($l \neq m$) and the diagonal or "population" terms ($l = m$) separately.

16.3.1 Off-Diagonal Relaxation

Consider the off-diagonal terms first. After discarding all terms in (16.10) not varying like $\hat{\sigma}_{lm}$ itself, we find

$$i\dot{\hat{\sigma}}_{lm} = \omega_{ml}\hat{\sigma}_{lm} - i\sum_{s>c}\sum_{p>a}\left(\Gamma_{scma}\delta_{pm}\delta_{ms}\delta_{ca} - \Gamma_{lcma}\delta_{sl}\delta_{cl}\delta_{ma}\delta_{pm} - \Gamma_{mcla}\delta_{pl}\delta_{sm}\delta_{mc}\delta_{al} - \Gamma_{scla}\delta_{pl}\delta_{sl}\delta_{ca}\right)\hat{\sigma}_{lm}$$

Now we will examine a detail left open so far. This has to do with the role of relaxation interactions not involving energy exchange with the reservoir, but rather phase change, i.e., elastic collision processes. They arise from terms associated with equality rather than inequality in the summation limits, e.g., $s = c$ and $p = a$. These terms are not included in the Hamiltonian (16.1). When these are included and separately taken into account, and the Kronecker δ-functions are evaluated, we find

$$\dot{\hat{\sigma}}_{lm} = -i\omega_{ml}\hat{\sigma}_{lm} - \left[\sum_{a\,(a<m)}\Gamma_{mama} + \sum_{a\,(a<l)}\Gamma_{lala} + \Gamma_{mmmm} + \Gamma_{llll} - \Gamma_{mmll} - \Gamma_{llmm}\right]\hat{\sigma}_{lm} \tag{16.11}$$

The rates with paired indices, namely Γ_{mmmm}, Γ_{llll}, Γ_{mmll} and Γ_{llmm} arise from the "dephasing" terms mentioned above. They satisfy

$$\Gamma_{mmmm} = \Gamma_{llll} \geq \Gamma_{mmll} = \Gamma_{llmm}$$

16.3.2 Diagonal Relaxation

The situation is similar, but not exactly the same, in the diagonal case $l = m$. Since $\hat{\sigma}_{mm}$ has a zero-frequency time oscillation, so any diagonal operator at all on the r.h.s. of (16.10) will remain "in phase"

with it, and cannot be discarded. We thus keep all diagonal operators, and the result is

$$\begin{aligned}\dot{\hat{\sigma}}_{mm} &= -\sum_{s>c}\sum_{p>a}\Gamma_{scpa}\left(\hat{\sigma}_{mp}\delta_{ms}\delta_{ca}+\hat{\sigma}_{pm}\delta_{ms}\delta_{ca}-\hat{\sigma}_{sp}\delta_{cm}\delta_{ma}-\hat{\sigma}_{ps}\delta_{mc}\delta_{am}\right)\\ &= -2\sum_{a\,(a<m)}\Gamma_{mama}\hat{\sigma}_{mm}+2\sum_{p\,(p>m)}\Gamma_{pmpm}\hat{\sigma}_{pp}\end{aligned} \tag{16.12}$$

This result is easy to interpret: The minus sign indicates that $\hat{\sigma}_{mm}$ decays at various rates Γ_{mama} into all the levels a below level m; the plus sign indicates that the population at level m increases with various rates Γ_{pmpm} due to decay into m from all levels p above it. Obviously, we have

$$\sum_m \dot{\hat{\sigma}}_{mm}=0$$

as the total population is conserved.

Note that the off-diagonal rate includes exactly the same decay coefficients as appear in the diagonal decay of either level l or level m. *Off-diagonal decay is at least half as fast as the sum of decay rates out of each of the two levels involved.* However, there is no contribution to the off-diagonal rates from the Γ_{pmpm} processes that cause level population to increase. That is, off-diagonal relaxation is directly affected by population decrease, but not at all by population increase.

16.4 Effects of the Noise Operator: Fluctuation-Dissipation Theorem

In the above discussion, we have neglected the contribution from the noise operators since their expectation values vanish. However, these noise operators exhibit non-trivial correlations with each other. Here we take a zero-temperature reservoir (the oscillator vacuum) and evaluate a two-time noise correlation function:

$$\begin{aligned}\langle\hat{n}_{sc}(t)\hat{n}^{\dagger}_{ps}(t')\rangle &= \sum_{\lambda,\mu}g(\lambda;s,c)\,g(\mu;p,a)\left\langle\hat{B}_{\lambda}(0)\hat{B}^{\dagger}_{\mu}(0)\right\rangle e^{-i\omega_{\lambda}t}\,e^{i\omega_{\mu}t}\\ &= \sum_{\lambda}g(\lambda;s,c)\,g(\mu;p,a)e^{-i\omega_{\lambda}(t-t')}\\ &= G_{scpa}(t-t')\\ &\approx \Gamma_{scpa}\,\delta(t-t')\end{aligned}$$

Therefore we see that the noise correlation, which characterize the reservoir fluctuations, is closely related to the dissipation of coherence in the atom. This is called the *fluctuation-dissipation theorem.*

16.5 Application to Two-Level Systems

For the system with only two levels labelled 1 and 2, we have

$$\begin{aligned}\dot{\hat{\sigma}}_{11} &= 2\Gamma_{2121}\,\hat{\sigma}_{22}=-\dot{\hat{\sigma}}_{22}\\ \dot{\hat{\sigma}}_{12} &= -\gamma_{12}\,\hat{\sigma}_{12}\end{aligned}$$

where $\gamma_{12}=\Gamma_{2121}+\Gamma_{2222}+\Gamma_{1111}-2\Gamma_{2211}$.

Chapter 17

Dissipative Processes in Schrödinger Picture: Master Equation

In realistic situations, dissipation is always present. The presence of dissipation usually makes an initially pure state become mixed state. Hence one has to use density operator to describe the system of interest. The equation governing the time evolution of the density operator is called the **Master Equation**.

Dissipation originates from the coupling to the environment (often called the reservoir) which comprises a much larger system or ensemble of states. This coupling is typically weak compared with couplings within the system but may have a rather significant effect on the system, but negligible effect on the reservoir. The effect of the dissipation do not, in general, depend on the precise form of the reservoir nor on the details of the coupling.

The total Hamiltonian including both the system and the reservoir reads

$$\mathcal{H} = \mathcal{H}_s + \mathcal{H}_r + \mathcal{H}_I$$

In the interaction picture, the system-reservoir interaction Hamiltonian takes the general form

$$\mathcal{H}_I(t) = i\hbar[\hat{s}^\dagger \hat{F}(t) - \hat{F}^\dagger(t)\hat{s}] \tag{17.1}$$

where $\hat{s}$ and $\hat{s}^\dagger$ respectively annihilate and create a quantum of energy $\hbar\omega$ in the system, and

$$\hat{F}(t) = -i \int W(\Delta) e^{-i\Delta t} b(\Delta) d\Delta$$

is the Langevin operator that characterize the effect of the reservoir. Usually we have $\langle \hat{F}(t) \rangle = \langle \hat{F}^\dagger(t) \rangle = 0$. Here we assume that the modes in the reservoir have a continuous spectrum characterized by the detuning Δ, annihilation (creation) operator $b(\Delta)$ ($b^\dagger(\Delta)$), and the system-reservoir coupling amplitude $W(\Delta)$.

17.1 Derivation of the Master Equation

Let ρ_T, ρ and ρ_r represent the total, system and reservoir density operators, respectively. Hence $\rho = \mathrm{Tr}_r \rho_T$. We consider the general class of problems for which the system and the reservoir are initially uncorrelated,

i.e., $\rho_T(0) = \rho(0) \otimes \rho_r(0)$. In the interaction picture, we have

$$\dot{\rho}_T(t) = -\frac{i}{\hbar}[\mathcal{H}_I(t), \rho_T(t)]$$

and the master equation for ρ is obtained by tracing over the reservoir degrees of freedom

$$\dot{\rho}(t) = -\frac{i}{\hbar}\mathrm{Tr}_r[\mathcal{H}_I(t), \rho_T(t)] \tag{17.2}$$

Equation (17.2) can be solved iteratively as

$$\rho(t) = \rho(0) + \sum_{n=1}^{\infty}\left(-\frac{i}{\hbar}\right)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \cdots \int_0^{t_{n-1}} dt_n \, \mathrm{Tr}_r\{[\mathcal{H}_I(t_1), [\mathcal{H}_I(t_2), \cdots [\mathcal{H}_I(t_n), \rho_T(t_n)]]]\} \tag{17.3}$$

Certainly in real calculations, we have to truncate the summation at finite n and let $\rho_T(t_n) = \rho_r(t_n) \otimes \rho(t_n)$. Since we have assumed that the reservoir is little changed by the system-reservoir coupling, so we have $\rho_r(t) = \rho_r(0)$. Then the $n = 1$ term involves only terms proportional to $\langle \hat{F}(t)\rangle$ and $\langle \hat{F}^\dagger(t)\rangle$, which are zero. Normally it suffices to calculate the second order $n = 2$ term.

Thus to second order, the master equation becomes

$$\begin{aligned}
\dot{\rho}(t) &= -\frac{1}{\hbar^2}\int_0^t dt' \, \mathrm{Tr}_r\{[\mathcal{H}_I(t), [\mathcal{H}_I(t'), \rho_T(t')]]\} \\
&= \int_0^t dt' \, \mathrm{Tr}_r\{[\hat{s}\hat{F}(t) - \hat{F}^\dagger(t)\hat{s}, [\hat{s}\hat{F}(t') - \hat{F}^\dagger(t')\hat{s}, \rho_r(0) \otimes \rho(t')]]\}
\end{aligned}$$

Expanding the commutators gives 16 terms. Using $\mathrm{Tr}_r[\hat{F}(t)\rho_r\hat{F}^\dagger(t')] = \mathrm{Tr}_r[\rho_r\hat{F}^\dagger(t')\hat{F}(t)] = \langle \hat{F}^\dagger(t')\hat{F}(t)\rangle$, we have

$$\begin{aligned}
\dot{\rho}(t) = \int_0^t dt' \Big\{ & [\hat{s}\rho(t')\hat{s}^\dagger - \hat{s}^\dagger\hat{s}\rho(t')]\langle \hat{F}(t)\hat{F}^\dagger(t')\rangle + [\hat{s}\rho(t')\hat{s}^\dagger - \rho(t')\hat{s}^\dagger\hat{s}]\langle \hat{F}(t')\hat{F}^\dagger(t)\rangle \\
& + [\hat{s}^\dagger\rho(t')\hat{s} - \hat{s}\hat{s}^\dagger\rho(t')]\langle \hat{F}^\dagger(t)\hat{F}(t')\rangle + [\hat{s}^\dagger\rho(t')\hat{s} - \rho(t')\hat{s}\hat{s}^\dagger]\langle \hat{F}^\dagger(t')\hat{F}(t)\rangle \\
& - [\hat{s}^\dagger\rho(t')\hat{s}^\dagger - \hat{s}^\dagger\hat{s}^\dagger\rho(t')]\langle \hat{F}(t)\hat{F}(t')\rangle - [\hat{s}^\dagger\rho(t')\hat{s}^\dagger - \rho(t')\hat{s}^\dagger\hat{s}^\dagger]\langle \hat{F}(t')\hat{F}(t)\rangle \\
& - [\hat{s}\rho(t')\hat{s} - \hat{s}\hat{s}\rho(t')]\langle \hat{F}^\dagger(t)\hat{F}^\dagger(t')\rangle - [\hat{s}\rho(t')\hat{s} - \rho(t')\hat{s}\hat{s}]\langle \hat{F}^\dagger(t')\hat{F}^\dagger(t)\rangle \Big\}
\end{aligned}$$

Under the Markov approximation, the Langevin operators have zero expectation values, i.e., $\langle \hat{F}\rangle = \langle \hat{F}^\dagger\rangle = 0$, and are δ-correlated, with

$$\begin{aligned}
\langle \hat{F}(t)\hat{F}^\dagger(t')\rangle &= \langle \hat{F}(t')\hat{F}^\dagger(t)\rangle = \frac{\Gamma}{2}(\bar{n}+1)\,\delta(t-t'), \quad \langle \hat{F}^\dagger(t)\hat{F}(t')\rangle = \langle \hat{F}^\dagger(t')\hat{F}(t)\rangle = \frac{\Gamma}{2}\bar{n}\,\delta(t-t') \\
\langle \hat{F}(t)\hat{F}(t')\rangle &= \langle \hat{F}(t')\hat{F}(t)\rangle = \frac{\Gamma}{2}M\,\delta(t-t'), \quad \langle \hat{F}^\dagger(t)\hat{F}^\dagger(t')\rangle = \langle \hat{F}^\dagger(t')\hat{F}^\dagger(t)\rangle = \frac{\Gamma}{2}M^*\,\delta(t-t')
\end{aligned}$$

with $\Gamma = 2\pi|W(0)|^2$, $\bar{n} = \langle b^\dagger(0)b(0)\rangle$ and $M = \langle b(0)b(0)\rangle$. From which, the master equation becomes

$$\begin{aligned}
\dot{\rho}(t) = & \frac{\Gamma}{2}(\bar{n}+1)[2\hat{s}\rho(t)\hat{s}^\dagger - \hat{s}^\dagger\hat{s}\rho(t) - \rho(t)\hat{s}^\dagger\hat{s}] + \frac{\Gamma}{2}\bar{n}[2\hat{s}^\dagger\rho(t)\hat{s} - \hat{s}\hat{s}^\dagger\rho(t) - \rho(t)\hat{s}\hat{s}^\dagger] \\
& - \frac{\Gamma}{2}M[2\hat{s}^\dagger\rho(t)\hat{s}^\dagger - \hat{s}^\dagger\hat{s}^\dagger\rho(t) - \rho(t)\hat{s}^\dagger\hat{s}^\dagger] - \frac{\Gamma}{2}M^*[2\hat{s}\rho(t)\hat{s} - \hat{s}\hat{s}\rho(t) - \rho(t)\hat{s}\hat{s}]
\end{aligned}$$

In general, the positivity of the density operator requires that $|M|^2 \leq \bar{n}(\bar{n}+1)$. In most cases, the reservoir is in a thermal equilibrium state, then $M = M^* = 0$, and $\bar{n}$ is the average number of the thermal quanta. For zero temperature, $\bar{n} = 0$ and the master equation is further simplified. A finite M indicates that the reservoir possesses phase sensitive correlations. This arises, for example, in the case of a squeezed reservoir state.

17.2 Examples

Here we give a few examples of the master equation.

17.2.1 Damped Cavity

Consider a cavity mode coupled to a zero-temperature thermal reservoir. The system operators are $\hat{s} = a$ and $\hat{s}^\dagger = a^\dagger$. For the reservoir, we have $\bar{n} = M = 0$. Then we can write the master equation as

$$\dot{\rho}(t) = \Gamma(\hat{J} + \hat{L})\rho(t)$$

where $\hat{J}$ and $\hat{L}$ are superoperators defined as

$$\begin{aligned} \hat{J}\rho(t) &= a\rho(t)a^\dagger \\ \hat{L}\rho(t) &= -\frac{1}{2}[a^\dagger a\rho(t) + \rho(t)a^\dagger a] \end{aligned}$$

The master equation can be formally solved as

$$\rho(t) = \exp[\Gamma t(\hat{J} + \hat{L})]\rho(0)$$

Using the property

$$[\hat{J}, \hat{L}]\rho = -\hat{J}\rho$$

we can disentangle the exponential operator to obtain either

$$\rho(t) = \exp(\Gamma t\hat{L})\, \exp\{[1 - \exp(-\Gamma t)]\hat{J}\}\, \rho(0) \tag{17.4}$$

or

$$\rho(t) = \exp\{[\exp(\Gamma t) - 1]\hat{J}\}\, \exp(\Gamma t\hat{L})\, \rho(0) \tag{17.5}$$

- If the cavity is initially in a coherent state, i.e., $\rho(0) = |\alpha\rangle\langle\alpha|$, we want to use (17.4), since $\rho(0)$ is an eigenstate of $\hat{J}$:

  $$\hat{J}\rho(0) = a|\alpha\rangle\langle\alpha|a^\dagger = |\alpha|^2\rho(0)$$

  Hence we have

  $$\rho(t) = \exp\{[1 - \exp(-\Gamma t)]|\alpha|^2\}\, \exp(\Gamma t\hat{L})\rho(0)$$

  In order to evaluate $\exp(\Gamma t\hat{L})\rho(0)$, we realize that

  $$\exp(\Gamma t\hat{L})\rho(0) = \exp(-\Gamma t a^\dagger a/2)|\alpha\rangle\langle\alpha|\exp(-\Gamma t a^\dagger a/2)$$

  Using the number state expansion for the coherent state, we have

  $$\begin{aligned} \exp(-\Gamma t a^\dagger a/2)|\alpha\rangle &= \exp(-|\alpha|^2/2)\sum_{n=0}^{\infty}\frac{[\alpha\exp(-\Gamma t/2)]^n}{\sqrt{n!}}|n\rangle \\ &= \exp\{-|\alpha|^2[1 - \exp(-\Gamma t)]/2\}|\alpha\exp(-\Gamma t/2)\rangle \end{aligned}$$

  Therefore, we have

  $$\rho(t) = |\alpha\exp(-\Gamma t/2)\rangle\langle\alpha\exp(-\Gamma t/2)|$$

  i.e., the cavity mode remains in a coherent state whose amplitude decays with rate $\Gamma/2$.

- If the cavity is initially in a number state, i.e., $\rho(0) = |n\rangle\langle n|$, we want to use (17.5), since now $\rho(0)$ is an eigenstate of $\hat{L}$:

$$\hat{L}\rho(0) = -\frac{1}{2}(a^\dagger a|n\rangle\langle n| + |n\rangle\langle n|a^\dagger a) = -n|n\rangle\langle n|$$

Hence we have

$$\rho(t) = \exp(-n\Gamma t)\, \exp\{[\exp(\Gamma t) - 1]\hat{J}\}\rho(0)$$

Consider $\exp(k\hat{J})$ as a power series in k giving

$$\exp(k\hat{J})\rho(0) = \sum_{l=0}^{\infty} \frac{k^l}{l!} a^l|n\rangle\langle n|(a^\dagger)^l = \sum_{l=0}^{\infty} \frac{k^l}{l!} \frac{n!}{(n-l)!}|n-l\rangle\langle n-l|$$

With $k = \exp(\Gamma t) - 1$, we have

$$\rho(t) = \sum_{l=0}^{\infty} [\exp(-\Gamma t)]^{n-1} \left[1 - \exp(-\Gamma t)\right]^l \frac{n!}{l!(n-l)!}|n-l\rangle\langle n-l|$$

corresponding to a mixed state with a binomial photon number probability distribution.

17.2.2 Two-Level-Atom

Consider a two-level atom with ground state $|g\rangle$ and excited state $|e\rangle$. Now the system operators are $\hat{s} = \hat{\sigma}_- = |g\rangle\langle e|$ and $\hat{s}^\dagger = \hat{\sigma}_+ = |e\rangle\langle g|$.

- Suppose the two-level-atom is couple to a thermal reservoir ($M = 0$). The master equation becomes

$$\dot{\rho}(t) = \frac{\Gamma}{2}(\bar{n}+1)(2|g\rangle\rho_{ee}\langle g| - |e\rangle\langle e|\rho - \rho|e\rangle\langle e|) + \frac{\Gamma}{2}\bar{n}(2|e\rangle\rho_{gg}\langle e| - |g\rangle\langle g|\rho - \rho|g\rangle\langle g|)$$

From which we have

$$\begin{aligned}
\dot{\rho}_{ee} &= -\dot{\rho}_{gg} = \Gamma(\bar{n}+1)\rho_{ee} - \Gamma\bar{n}\rho_{gg} \\
\dot{\rho}_{eg} &= -\frac{\Gamma}{2}(2\bar{n}+1)\rho_{eg}
\end{aligned}$$

which can be solved to give

$$\begin{aligned}
\rho_{ee}(t) &= 1 - \rho_{gg}(t) = \rho_{ee}(0)e^{-\Gamma(2\bar{n}+1)t} + \frac{\bar{n}}{2\bar{n}+1}\left[1 - e^{-\Gamma(2\bar{n}+1)t}\right] \\
\rho_{eg}(t) &= \rho_{eg}(0)e^{-\Gamma(\bar{n}+1/2)t}
\end{aligned}$$

which shows that the atom evolves towards thermodynamic equilibrium with the reservoir.

- Suppose the atom is driven by a resonant laser field and coupled to vacuum ($\bar{n} = 0$) through spontaneous emission. The atom-laser interaction is described by the Hamiltonian

$$H_{aL} = \frac{\hbar\Omega}{2}(\hat{\sigma}_+ + \hat{\sigma}_-)$$

Now we have to add the term $-(i/\hbar)[H_{aL}, \rho(t)]$ at the r.h.s. of the master equation (assuming zero detuning). Define

$$\hat{\sigma}_1 = \hat{\sigma}_+ + \hat{\sigma}_-, \quad \hat{\sigma}_2 = i(\hat{\sigma}_- - \hat{\sigma}_+), \quad \hat{\sigma}_3 = |e\rangle\langle e| - |g\rangle\langle g|$$

we can write

$$\rho(t) = \frac{1}{2}[\mathbf{1} + u(t)\hat{\sigma}_1 + v(t)\hat{\sigma}_2 + w(t)\hat{\sigma}_3]$$

with $\mathbf{1} = |e\rangle\langle e| + |g\rangle\langle g|$ being the identity matrix and u, v, w are the expectation values of $\hat{\sigma}_{1,2,3}$, respectively. We can now derive the equation of motion for w as

$$\begin{aligned}
\dot{w}(t) &= \mathrm{Tr}[\hat{\sigma}_3\dot{\rho}(t)] \\
&= -i\frac{\Omega}{2}\mathrm{Tr}[\hat{\sigma}_3\hat{\sigma}_1\rho(t) - \hat{\sigma}_3\rho(t)\hat{\sigma}_1] + \frac{\Gamma}{2}\mathrm{Tr}[2\hat{\sigma}_3\hat{\sigma}_-\rho(t)\hat{\sigma}_+ - \hat{\sigma}_3\hat{\sigma}_+\hat{\sigma}_-\rho(t) - \hat{\sigma}_3\rho(t)\hat{\sigma}_+\hat{\sigma}_-] \\
&= -i\frac{\Omega}{2}\mathrm{Tr}\{[\hat{\sigma}_3, \hat{\sigma}_1]\rho(t)\} - 2\Gamma\mathrm{Tr}[\hat{\sigma}_+\hat{\sigma}_-\rho(t)] \\
&= \Omega v(t) - \Gamma[w(t) + 1]
\end{aligned}$$

Similarly, we can find

$$\begin{aligned}
\dot{u}(t) &= -\frac{\Gamma}{2}u(t) \\
\dot{v}(t) &= -\Omega w(t) - \frac{\Gamma}{2}v(t)
\end{aligned}$$

These are just the Optical Bloch Equations.

17.3 Monte Carlo Wave Function Method

If a relevant Hilbert space of the quantum system has a dimension N that is much larger than 1, the number of variables involved in solving the Master equation for the density matrices is $\sim N^2$. The computation can become very time-consuming, or simply unpractical. To overcome this, *Monte Carlo wave function method* (MCWF), or equivalently the *quantum trajectory method*, is developed to study the system dynamics. In such methods, the number of variables involved is $\sim N$.

To see the equivalence between the MCWF method and the Master equation, we note that the equation obeyed by the system density operator is

$$\dot{\rho} = \frac{1}{i\hbar}[\mathcal{H}_s, \rho] + \mathcal{L}_{\mathrm{relax}}(\rho) \tag{17.6}$$

where $\mathcal{L}_{\mathrm{relax}}$ represents the relaxation superoperator, acting on the density operator ρ. In most quantum optics problems involving dissipation, the relaxation has the form

$$\mathcal{L}_{\mathrm{relax}}(\rho) = -\frac{1}{2}\sum_m \left(C_m^\dagger C_m\rho + \rho C_m^\dagger C_m\rho\right) + \sum_m C_m\rho C_m^\dagger \tag{17.7}$$

The first two terms at the r.h.s. of (17.7) can be accounted for if we define the *non-Hermitian* Hamiltonian as

$$\mathcal{H}_{\mathrm{eff}} = \mathcal{H}_s - i\frac{\hbar}{2}\sum_m C_m^\dagger C_m$$

The last term at the r.h.s. of (17.7) represents some kind of projection as can be seen if we take $\rho = |\psi\rangle\langle\psi|$, then this term corresponds to a projection operator $\sum_m P_{|\psi'\rangle_m} = \sum_m |\psi'\rangle_m{}_m\langle\psi'|$ with $|\psi'\rangle_m = C_m|\psi\rangle$. Hence this term corresponds to a *quantum jump*. If the quantum jump indeed happens, then the system

wave function immediately following the jump will be projected into one of the $|\psi'\rangle_m$'s with some probability law.

Suppose at time t, the system is in a state with the normalized wave function $|\phi(t)\rangle$. The procedure of the MCWF method to find the wave function $|\phi(t+\delta t)\rangle$ then takes the following steps:

- First we calculate the probability for the quantum jump to occur, which is given by

$$\delta p = \delta t\,(i/\hbar)\langle\phi(t)|(\mathcal{H}_{\rm eff}-\mathcal{H}_{\rm eff}^\dagger)|\phi(t)\rangle = \sum_m \delta p_m \tag{17.8}$$

$$\delta p_m = \delta t\,\langle\phi(t)|C_m^\dagger C_m|\phi(t)\rangle \tag{17.9}$$

  δt must be chosen such that $\delta p \ll 1$.

- Then a random number ϵ, uniformly distributed between 0 and 1, is chosen, and we compare ϵ with δp.

  – If $\epsilon > \delta p$ which occurs for most of the time since $\delta p \ll 1$, then the quantum jump does not occur, and the system evolves according to $\mathcal{H}_{\rm eff}$, hence we have

$$|\phi(t+\delta t)\rangle = \mathcal{N} e^{-i\mathcal{H}_{\rm eff}\delta t/\hbar}|\phi(t)\rangle$$

  where $\mathcal{N}$ is a normalization constant which is necessary since $\mathcal{H}_{\rm eff}$ is not Hermitian, hence the propagator $e^{-i\mathcal{H}_{\rm eff}\delta t/\hbar}$ does not preserve the norm of the wave function. It can be calculated as

$$\frac{1}{\mathcal{N}} = \sqrt{\langle\phi(t)|e^{i(\mathcal{H}_{\rm eff}^\dagger-\mathcal{H}_{\rm eff})\delta t/\hbar}|\phi(t)\rangle} = \sqrt{\langle\phi(t)|e^{-\delta t\sum_m C_m^\dagger C_m}|\phi(t)\rangle}$$

  To first order in δt, we have

$$|\phi(t+\delta t)\rangle = \frac{1}{\sqrt{1-\delta p}}(1-i\mathcal{H}_{\rm eff}\delta t/\hbar)|\phi(t)\rangle$$

  – If $\epsilon \le \delta p$, then the quantum jump does occur, and the system wave function becomes

$$|\phi(t+\delta t)\rangle_m = \mathcal{N}' C_m|\phi(t)\rangle$$

  with a probability $\Pi_m = \delta p_m/\delta p$ and $\mathcal{N}' = \left(\langle\phi(t)|C_m^\dagger C_m|\phi(t)\rangle\right)^{-1/2} = \sqrt{\delta t/\delta p_m}$ is again the normalization constant.

Now the equivalence of the MCWF with the Master equation can be proved as follows. The density operator at time t is given by $\rho(t) = |\phi(t)\rangle\langle\phi(t)|$, and at time $t+\delta t$, the averaged $\rho(t+\delta t)$ over the evolution caused by many different values of the random number ϵ is

$$\begin{aligned}\rho(t+\delta t) &= (1-\delta p)|\phi(t+\delta t)\rangle\langle\phi(t+\delta t)| + \delta p\sum_m \Pi_m|\phi(t+\delta t)\rangle_{mm}\langle\phi(t+\delta t)| \\ &= \rho(t) + \frac{\delta t}{i\hbar}[\mathcal{H}_s,\rho(t)] + \delta t\mathcal{L}_{\rm relax}[\rho(t)]\end{aligned}$$

which agrees with the Master equation (17.6).

17.3.1 Example: Application to Two-Level Atom

Now let us apply the MCWF to a two-level atom driven by a laser field subject to spontaneous emission. The Hamiltonian and the Master equation are given by

$$\begin{aligned} H_s &= -\hbar\Delta\hat{\sigma}_+\hat{\sigma}_- + \frac{1}{2}\hbar\Omega(\hat{\sigma}_+ + \hat{\sigma}_-) \\ \dot{\rho} &= \frac{1}{i\hbar}[H_s, \rho] - \frac{\Gamma}{2}(\hat{\sigma}_+\hat{\sigma}_-\rho + \rho\hat{\sigma}_+\hat{\sigma}_-) + \Gamma\hat{\sigma}_-\rho\hat{\sigma}_+ \end{aligned}$$

Hence the relaxation terms in the Master equation indeed has the form of (17.7), with only one $C_m = \sqrt{\Gamma}\hat{\sigma}_-$. The terms $\hat{\sigma}_+\hat{\sigma}_-\rho + \rho\hat{\sigma}_+\hat{\sigma}_-$ is responsible for the decay of coherence (off-diagonal density matrix elements) and that of the excited population, while the quantum jumping term $\hat{\sigma}_-\rho\hat{\sigma}_+$ corresponds to the spontaneous emission of a photon and the state being projected onto the ground state. The effective non-Hermitian Hamiltonian is then given by

$$H_{\text{eff}} = H_s - i\frac{\hbar\Gamma}{2}\hat{\sigma}_+\hat{\sigma}_-$$

The procedure of the simulation is as follows. Given the atomic wave function $\psi(t)\rangle = a_g(t)|g\rangle + a_e(t)|e\rangle$, the wave function at $t + \delta t$ is determined through the following steps

- First we calculate the probability of spontaneous emission between t and $t + \delta t$ which is given by

$$\delta p = \delta t\Gamma\langle\psi(t)|\hat{\sigma}^+\hat{\sigma}^-|\psi(t)\rangle = \delta t\Gamma|a_e(t)|^2$$

- A random number ϵ is chosen to determine whether or not the spontaneous emission occurs.
  - If $\epsilon > \delta p$, then the photon is not emitted, the state evolves as

$$|\psi(t+\delta t)\rangle = \mathcal{N}e^{-iH_{\text{eff}}\delta t/\hbar}\,|\psi(t)\rangle$$

  - If $\epsilon \leq \delta p$, then $|\psi(t+\delta t)\rangle = |g\rangle$

Chapter 18

Dissipative Processes in Heisenberg Picture: Quantum Langevin Equations and Quantum Regression Theorem

To gain more insight into the system-reservoir interaction, we now treat the same problem in the Heisenberg picture. In this treatment, the reservoir operators can be interpreted similarly to Langevin forces in classical statistical mechanics. These *quantum noise operators* are the source of both fluctuations and irreversible dissipation of energy from the system to the reservoir. Here we will study a specific example of the interaction between a two-level atom and the free space electromagnetic field.

18.1 Quantum Langevin Equations

The system operators are then

$$\sigma_+ = |e\rangle\langle g|, \quad \sigma_- = |g\rangle\langle e|, \quad \sigma_3 = |e\rangle\langle e| - |g\rangle\langle g|$$

and $\sigma_1 = \sigma_+ + \sigma_-$, $\sigma_2 = i(\sigma_- - \sigma_+)$. The reservoir is characterized by the annihilation and creation operators $b(\Delta, t)$ and $b^\dagger(\Delta, t)$. The interaction Hamiltonian is given by

$$H = \hbar \int \Delta b^\dagger(\Delta, t) b(\Delta, t) d\Delta + \hbar \int \left[W(\Delta)\sigma_+(t) b(\Delta, t) + W^*(\Delta) b^\dagger(\Delta, t)\sigma_-(t) \right] d\Delta$$

From which we can derive the Heisenberg equations of motion as

$$\dot{\sigma}_- = i\int W(\Delta)\sigma_3(t)b(\Delta,t)d\Delta \tag{18.1}$$

$$\dot{\sigma}_+ = -i\int W^*(\Delta)b^\dagger(\Delta,t)\sigma_3(t)d\Delta \tag{18.2}$$

$$\dot{\sigma}_3 = -2i\int \left[W(\Delta)\sigma_+(t)b(\Delta,t) - W^*(\Delta)b^\dagger(\Delta,t)\sigma_-(t)\right]d\Delta \tag{18.3}$$

$$\dot{b}(\Delta,t) = -i\Delta b(\Delta,t) - iW^*(\Delta)\sigma_-(t) \tag{18.4}$$

We can formally integrate the last equation to solve for $b(\Delta,t)$ as:

$$b(\Delta,t) = b(\Delta,0)e^{-i\Delta t} - iW^*(\Delta)\int_0^t e^{-i\Delta(t-t')}\sigma_-(t')dt'$$

Put the above solution back to the equations for system operators, we have (we only use σ_- as an example)

$$\dot{\sigma}_-(t) = i\int W(\Delta)\sigma_3(t)b(\Delta,0)e^{-i\Delta t}d\Delta + \int_0^t dt'\int d\Delta\,|W(\Delta)|^2 e^{-i\Delta(t-t')}\sigma_3(t)\sigma_-(t') \tag{18.5}$$

To proceed further, we take the Markov approximation, i.e., assume that $K(t-t') = \int d\Delta\,|W(\Delta)|^2 e^{-i\Delta(t-t')}$ to be sharply peaked at $t=t'$. As a result, we can replace the argument t' in $\sigma_-(t')$ of the double integral by t and hence take it out of the integral [Note that $\sigma_3(t)\sigma_-(t) = -\sigma_-(t)$]. This is equivalent to assuming that $\sigma_-(t)$ does not dependent on its previous values. In the extreme limit, we can take $K(t-t')$ to be proportional to a δ-function: $K(t-t') = \Gamma\delta(t-t')$ with $\Gamma = 2\pi|W(0)|^2$. Then we can rewrite (18.5) as

$$\dot{\sigma}_-(t) = -\frac{\Gamma}{2}\sigma_-(t) - \sigma_3(t)\hat{F}(t) \tag{18.6}$$

where $\hat{F}(t) = -i\int W(\Delta)b(\Delta,0)e^{-i\Delta t}d\Delta$ is the *Langevin operator*, sometimes also called the *noise operator* or *fluctuation operator*, which depends on the field operator at the initial time.

The Langevin operators can be evaluated at least in principle, but their explicit form is not needed within the Markov approximation and with the quantum regression theorem (see later). This remarkable result eliminates the need for considerable algebra. All we need to know about the noise operators is that they have zero expectation values, i.e., $\langle\hat{F}\rangle = \langle\hat{F}^\dagger\rangle = 0$, and are δ-correlated, with

$$\langle\hat{F}(t)\hat{F}^\dagger(t')\rangle = \langle\hat{F}(t')\hat{F}^\dagger(t)\rangle = \frac{\Gamma}{2}(\bar{n}+1)\,\delta(t-t'), \quad \langle\hat{F}^\dagger(t)\hat{F}(t')\rangle = \langle\hat{F}^\dagger(t')\hat{F}(t)\rangle = \frac{\Gamma}{2}\bar{n}\,\delta(t-t')$$

$$\langle\hat{F}(t)\hat{F}(t')\rangle = \langle\hat{F}(t')\hat{F}(t)\rangle = \frac{\Gamma}{2}M\,\delta(t-t'), \quad \langle\hat{F}^\dagger(t)\hat{F}^\dagger(t')\rangle = \langle\hat{F}^\dagger(t')\hat{F}^\dagger(t)\rangle = \frac{\Gamma}{2}M^*\,\delta(t-t')$$

with $\bar{n} = \langle b^\dagger(0)b(0)\rangle$ and $M = \langle b(0)b(0)\rangle$.

Similarly we can have

$$\dot{\sigma}_+(t) = -\frac{\Gamma}{2}\sigma_+(t) - \hat{F}^\dagger(t)\sigma_3(t) \tag{18.7}$$

$$\dot{\sigma}_3(t) = -\Gamma[\sigma_3(t)+1] + 2\sigma_+(t)\hat{F}(t) + 2\hat{F}^\dagger(t)\sigma_-(t) \tag{18.8}$$

These equations are called the *quantum Langevin equations*

The properties of atom at any given time are described by the expectation values of the atomic operators. It is sufficient, therefore, to obtain equations of motion for $\langle\sigma_\pm(t)\rangle$ and $\langle\sigma_3(t)\rangle$. In order to do so, we have to deal with the expectation value of the products of Langevin and atomic operators. To simply decorrelate the

products as $\langle\sigma_3(t)\hat{F}(t)\rangle = \langle\sigma_3(t)\rangle\langle\hat{F}(t)\rangle = 0$ obviously doesn't make sense as it neglects part of the influence of the field on the atom, and the resulting equations are incorrect. To overcome this, we first formally integrate (18.6), (18.7) and (18.8) to give

$$\sigma_-(t) = e^{-\Gamma t/2}\left[\sigma_-(0) - \int_0^t e^{\Gamma t'/2}\sigma_3(t')\hat{F}(t')dt'\right] \tag{18.9}$$

$$\sigma_+(t) = e^{-\Gamma t/2}\left[\sigma_+(0) - \int_0^t e^{\Gamma t'/2}\hat{F}^\dagger(t')\sigma_3(t')dt'\right] \tag{18.10}$$

$$\sigma_3(t)+1 = e^{-\Gamma t}\left[\sigma_3(0)+1+2\int_0^t e^{\Gamma t'}[\sigma_+(t')\hat{F}(t') + \hat{F}^\dagger(t')\sigma_-(t')]dt'\right] \tag{18.11}$$

and to proceed to the next order of iteration. Substituting these into (18.6), (18.7) and (18.8) and then taking the expectation value of each equation gives

$$\langle\dot{\sigma}_-(t)\rangle = -\frac{\Gamma}{2}\langle\sigma_-(t)\rangle - 2\int_0^t e^{-\Gamma(t-t')}[\langle\sigma_+(t')\hat{F}(t')\hat{F}(t)\rangle + \langle\hat{F}^\dagger(t')\hat{F}(t)\sigma_-(t')\rangle]dt'$$

$$\langle\dot{\sigma}_+(t)\rangle = -\frac{\Gamma}{2}\langle\sigma_+(t)\rangle - 2\int_0^t e^{-\Gamma(t-t')}[\langle\hat{F}^\dagger(t)\hat{F}^\dagger(t')\sigma_-(t')\rangle + \langle\sigma_+(t')\hat{F}^\dagger(t)\hat{F}(t')\rangle]dt'$$

$$\langle\dot{\sigma}_3(t)\rangle = -\Gamma[\langle\sigma_3(t)\rangle+1] - 2\int_0^t e^{-\Gamma(t-t')/2}[\langle\hat{F}^\dagger(t')\sigma_3(t')\hat{F}(t)\rangle + \langle\hat{F}^\dagger(t)\sigma_3(t')\hat{F}(t')\rangle]dt'$$

At this stage, we can decorrelate atomic and Langevin operators, and using the properties of the Langevin operators, we have

$$\langle\dot{\sigma}_-(t)\rangle = -\frac{\Gamma}{2}(2\bar{n}+1)\langle\sigma_-(t)\rangle - \Gamma M\langle\sigma_+(t)\rangle \tag{18.12}$$

$$\langle\dot{\sigma}_+(t)\rangle = -\frac{\Gamma}{2}(2\bar{n}+1)\langle\sigma_+(t)\rangle - \Gamma M^*\langle\sigma_-(t)\rangle \tag{18.13}$$

$$\langle\dot{\sigma}_3(t)\rangle = -\Gamma[(2\bar{n}+1)\langle\sigma_3(t)\rangle+1] \tag{18.14}$$

From which we also have (assume M to be real)

$$\langle\dot{\sigma}_1(t)\rangle = -\frac{\Gamma}{2}(2\bar{n}+1+2M)\langle\sigma_1(t)\rangle$$

$$\langle\dot{\sigma}_2(t)\rangle = -\frac{\Gamma}{2}(2\bar{n}+1-2M)\langle\sigma_2(t)\rangle$$

18.2 Quantum Regression Theorem

In many circumstances, we need to calculate the two-time correlation function such as $\langle\sigma_+(t)\sigma_-(t')\rangle$ with $t > t'$. This can be done using the *quantum regression theorem*, which simply states that the expectation value of the two-time correlation function $\langle\sigma_i(t)\sigma_j(t')\rangle$ satisfies the same equation of motion as the single-time $\langle\sigma_i(t)\rangle$ does. Here is the proof.

From (18.7), we have

$$\frac{d}{dt}\langle\sigma_+(t)\sigma_-(t')\rangle = -\frac{\Gamma}{2}\langle\sigma_+(t)\sigma_-(t')\rangle - \langle\hat{F}^\dagger(t)\sigma_3(t)\sigma_-(t')\rangle$$

Inserting the formal solution of $\sigma_3(t)$ in (18.11), the last term becomes

$$\langle\hat{F}^\dagger(t)\sigma_3(t)\sigma_-(t')\rangle = \left\langle\hat{F}^\dagger(t)[-1+e^{-\Gamma t}(\sigma_3(0)+1)]\sigma_-(t')\right\rangle + 2e^{-\Gamma t}\left\langle\hat{F}^\dagger(t)\int_0^t e^{\Gamma\tau}[\sigma_+(\tau)\hat{F}(\tau)+\hat{F}^\dagger(\tau)\sigma_-(\tau)]d\tau\,\sigma_-(t')\right\rangle$$

Since $t > t'$ and the atomic operator cannot depend on the noise operator at a future time, so the first term at the r.h.s. vanishes. Decorrelate the atomic and Langevin operators in the second term, we have

$$\langle \hat{F}^\dagger(t)\sigma_3(t)\sigma_-(t')\rangle = \Gamma\bar{n}\langle\sigma_+(t)\sigma_-(t')\rangle + \Gamma M^*\langle\sigma_-(t)\sigma_-(t')\rangle$$

Hence we have

$$\frac{d}{dt}\langle\sigma_+(t)\sigma_-(t')\rangle = -\frac{\Gamma}{2}(2\bar{n}+1)\langle\sigma_+(t)\sigma_-(t')\rangle - \Gamma M^*\langle\sigma_-(t)\sigma_-(t')\rangle$$

Compared with (18.13), we see that the two-time correlation function obeys the same equation as $\langle\sigma_+(t)\rangle$, thus establishing the quantum regression theorem for the two-level atom.

Chapter 19

Optical Bloch Equations

19.1 Semiclassical Theory of Atom-Photon Interaction

In previous lectures, we have studied the interaction between atom and light field. We have treated both the atom and the light quantum mechanically. In many situations, however, the quantum effect of light is negligible. Under such circumstances, a semiclassical theory suffices, in which the atom is still quantized (i.e., having discretized energy levels), while the light is treated classically (i.e., obeying Maxwell equations). Obviously, there are situations when the semiclassical theory is invalid. For example, the semiclassical theory cannot correctly describe the quantum beats phenomenon since it neglects the quantum correlations between atomic operators and the light field which is the key to understanding quantum beats.

Here we'll study the semiclassical theory and we'll start from the interaction between a two-level atom and a classical light field.

19.2 Operator Physics for a Two-Level Atom

We've already introduced two-level atom in the study of JC model. The states for a two-level atom: $|g\rangle$ and $|e\rangle$. They are assumed to have opposite parity (hence dipole transition is allowed) and orthogonal to each other. From these one can construct four independent operators:

$$|g\rangle\langle g|, \;\; |g\rangle\langle e|, \;\; |e\rangle\langle g|, \;\; |e\rangle\langle e|,$$

which form a complete basis. Any arbitrary operator, $\hat{O}$, can then be expanded onto this basis as

$$\hat{O} = O_{gg}\hat{\sigma}_{gg} + O_{ge}\hat{\sigma}_{ge} + O_{eg}\hat{\sigma}_{eg} + O_{ee}\hat{\sigma}_{ee}$$

where $\hat{\sigma}_{ij} = |i\rangle\langle j|$, and $O_{ij} = \langle i|\hat{O}|j\rangle$. In particular, the dipole operator $\hat{\mathbf{d}} = e\hat{\mathbf{r}}$ can be expressed as

$$\hat{\mathbf{d}} = \mathbf{d}_{ge}\hat{\sigma}_{ge} + \mathbf{d}_{eg}\hat{\sigma}_{eg}$$

where we have used the property that states $|g\rangle$ and $|e\rangle$ have opposite parity such that $\langle g|\hat{\mathbf{r}}|g\rangle = \langle e|\hat{\mathbf{r}}|e\rangle = 0$.

19.3 Feynman-Bloch Vector and Optical Bloch Equations

Assume $\mathbf{d}_{eg} = \mathbf{d}_{ge} = \mathbf{d}$, the total Hamiltonian under the dipole approximation is:

$$H = \hbar\omega_0\hat{\sigma}_{ee} - \hat{\mathbf{d}}\cdot\mathbf{E} = \hbar\omega_0\hat{\sigma}_{ee} - \mathbf{d}\cdot\mathbf{E}(\hat{\sigma}_{ge} + \hat{\sigma}_{eg})$$

Using

$$i\hbar\dot{\hat{O}} = [\hat{O}, H],$$

the equations of motion in Heisenberg picture for $\sigma_{ij} = \langle\hat{\sigma}_{ij}\rangle$ are (note that the equations of motion for operators $\hat{\sigma}_{ij}$ are linear, their respective expectation values σ_{ij} obey exactly the same equations.)

$$\begin{aligned}
i\hbar\dot{\sigma}_{gg} &= -\mathbf{d}\cdot\mathbf{E}(\sigma_{ge} - \sigma_{eg}) \\
i\hbar\dot{\sigma}_{ee} &= \mathbf{d}\cdot\mathbf{E}(\sigma_{ge} - \sigma_{eg}) \\
i\hbar\dot{\sigma}_{eg} &= -\hbar\omega_0\sigma_{eg} - \mathbf{d}\cdot\mathbf{E}(\sigma_{ee} - \sigma_{gg}) \\
i\hbar\dot{\sigma}_{ge} &= \hbar\omega_0\sigma_{ge} + \mathbf{d}\cdot\mathbf{E}(\sigma_{ee} - \sigma_{gg})
\end{aligned}$$

These are called the *optical Bloch equations* (OBEs). Instead of σ_{ij}, it is sometimes more convenient to use the following 4 real operators

$$S_0 = \sigma_{gg} + \sigma_{ee}, \quad \text{total probability} \tag{19.1}$$

$$S_1 = \sigma_{ge} + \sigma_{eg}, \quad \text{dipole moment} \tag{19.2}$$

$$S_2 = i(\sigma_{ge} - \sigma_{eg}), \quad \text{dipole current} \tag{19.3}$$

$$S_3 = \sigma_{ee} - \sigma_{gg}, \quad \text{population inversion} \tag{19.4}$$

That S_2 represents the dipole current can be seen from the following. For a non-interacting atom, the Heisenberg equation of motion for its position operator is given by

$$\frac{d}{dt}\hat{\mathbf{r}} = \frac{\hat{\mathbf{p}}}{m}$$

At the same time, we can compute $d\hat{\mathbf{r}}/dt$ from the H_A commutator:

$$i\hbar\frac{d}{dt}\hat{\mathbf{r}} = [\hat{\mathbf{r}}, H_A] = \hbar\omega_0[\hat{\mathbf{r}}, |e\rangle\langle e|] = \hbar\omega_0(\mathbf{r}_{ge}\hat{\sigma}_{ge} - \mathbf{r}_{eg}\hat{\sigma}_{eg})$$

Using the phase convention such that $\mathbf{r}_{eg} = \mathbf{r}_{ge}$ and using $\mathbf{d} = e\mathbf{r}_{eg}$, we have

$$e\frac{\mathbf{p}}{m} = -i\omega_0\mathbf{d}(\sigma_{ge} - \sigma_{eg}) = -\omega_0\mathbf{d}S_2$$

Since charge times velocity ($e\mathbf{p}/m$) is called current, we see that indeed S_2 represents the dipole current.

Using S's, the OBEs take the form

$$\dot{S}_0 = 0 \tag{19.5}$$

$$\dot{S}_1 = -\omega_0 S_2 \tag{19.6}$$

$$\dot{S}_2 = \omega_0 S_1 + \frac{2\mathbf{d}\cdot\mathbf{E}}{\hbar}S_3 \tag{19.7}$$

$$\dot{S}_3 = -\frac{2\mathbf{d}\cdot\mathbf{E}}{\hbar}S_2 \tag{19.8}$$

The first of these equation is a consequence of conservation of total probability $S_0 = \sigma_{ee} + \sigma_{gg} = 1$. Define the Feynman-Bloch (FB) vector

$$\mathbf{S} = [S_1,\, S_2,\, S_3]$$

the last three equations of motion can be combined to give

$$\dot{\mathbf{S}} = \mathbf{\Omega}_{\rm opt} \times \mathbf{S} \tag{19.9}$$

where the vector

$$\mathbf{\Omega}_{\rm opt} = \left[-\frac{2\mathbf{d}\cdot\mathbf{E}}{\hbar},\, 0,\, \omega_0\right]$$

represents an effective magnetic field if we think of $\mathbf{S}$ as a magnetization vector.

The vector $\mathbf{S}$ was first constructed to study the nuclear spin motion in a magnetic field. Feynman was the first one to generalize it to a generic two-level system.

19.4 Main Motion of FB Vector

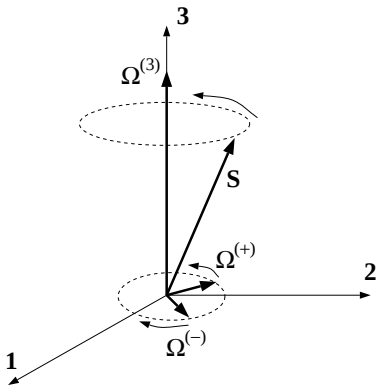

Figure 19-1: Motion of the Feynman-Bloch vector.

The motion of the FB vector yields everything about the time dependence of dipole moment, dipole current and population inversion. For typical parameters, $\omega_0 \gg |2\mathbf{d}\cdot\mathbf{E}/\hbar|$, so $\mathbf{\Omega}_{\rm opt} \approx \omega_0\hat{3}$ (designate by $\hat{1}$, $\hat{2}$ and $\hat{3}$ the fixed unit vectors of the three-dimensional coordinate system). Hence, from Eq. (19.9), the "main motion" of $\mathbf{S}$ is then simply constant precession about axis-$\hat{3}$.

Now, for a monochromatic light field,

$$\mathbf{E}(\mathbf{r},t) = \mathbf{E}_0\left(e^{-i\omega t} + c.c.\right) = 2\mathbf{E}_0\cos\omega t$$

we can decompose $\mathbf{\Omega}_{\rm opt}$ as

$$\mathbf{\Omega}_{\rm opt} = \mathbf{\Omega}^{(3)} + \mathbf{\Omega}^{(+)} + \mathbf{\Omega}^{(-)}$$

where

$$\mathbf{\Omega}^{(3)} = \omega_0\,\hat{3} \tag{19.10}$$

$$\mathbf{\Omega}^{(\pm)} = -R\,(\cos\omega t\,\hat{1} \pm \sin\omega t\,\hat{2}) \tag{19.11}$$

where $R = 2\mathbf{d} \cdot \mathbf{E}_0/\hbar$ is the so-called Rabi frequency.

We can see that $\mathbf{\Omega}^{(\pm)}$ rotate in the $\hat{1}$-$\hat{2}$ plane in opposite directions. $\mathbf{\Omega}^{(+)}$ rotates nearly in phase (co-rotating) with the main motion of $\mathbf{S}$, while $\mathbf{\Omega}^{(-)}$ (counter-rotating) rotates in the opposite direction from the main motion. We expect $\mathbf{S}$ to see a very rapidly alternating effect from $\mathbf{\Omega}^{(-)}$ and a persistent effect from $\mathbf{\Omega}^{(+)}$. Nearly synchronous rotation in the same direction as the main motion is called a co-rotating wave, and opposite rotation is called a counter-rotating wave. The Rotating Wave Approximation (RWA) results from the neglect of all effects of the counter-rotating terms.

19.5 Rotating Wave Approximation and RWA Equations

Taking the hint from the discussion above, let us define a rotating coordinate with rotating axis $\hat{3}$:

$$\mathbf{e}_1(t) = \cos\omega t\,\hat{1} + \sin\omega t\,\hat{2} \tag{19.12}$$

$$\mathbf{e}_2(t) = -\sin\omega t\,\hat{1} + \cos\omega t\,\hat{2} \tag{19.13}$$

$$\mathbf{e}_3 = \hat{3} \tag{19.14}$$

Key advantage of rotating frame: automatically separates time scales, allowing more detailed examination of slow but significant changes.

Now we want to derive the new equations of motion, i.e., the counterpart of Eq. (19.9) in the rotating frame. First, we decompose $\mathbf{S}$ in the new frame as

$$\mathbf{S} = u\,\mathbf{e}_1(t) + v\,\mathbf{e}_2(t) + w\mathbf{e}_3$$

where

$$\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} \cos\omega t & \sin\omega t & 0 \\ -\sin\omega t & \cos\omega t & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} S_1 \\ S_2 \\ S_3 \end{pmatrix}$$

Second, we want to decompose $\mathbf{\Omega}_{\text{opt}}$. Using

$$\mathbf{\Omega}^{(3)} = \omega_0\,\mathbf{e}_3 \tag{19.15}$$

$$\mathbf{\Omega}^{(+)} = -R\mathbf{e}_1 \tag{19.16}$$

$$\mathbf{\Omega}^{(-)} = -R\left(\cos 2\omega t\,\mathbf{e}_1 - \sin 2\omega t\,\mathbf{e}_2\right) \tag{19.17}$$

As one can see, $\mathbf{\Omega}^{(-)}$ represents terms rotating at frequency 2ω. These fast rotating terms average quickly to zero. This is the reason that we can neglect these double-frequency terms. This is called the Rotating Wave Approximation (RWA).

Now we seem to be ready to convert Eq. (19.9) into the rotating frame. But before doing so, we should remind ourselves about the Coriolis effect from classical mechanics: the rate of change of a vector $\mathbf{V}$ in a rotating frame is the rate of change of $\mathbf{V}$ in the original fixed frame minus a Coriolis term, which is given by $\omega\hat{a} \times \mathbf{V}$, where $\hat{a}$ is the unit vector in the direction of the axis of rotation and ω is the rate of rotation. Therefore,

$$\left(\dot{\mathbf{S}}\right)_{\text{rot}} = \left(\dot{\mathbf{S}}\right)_{\text{fixed}} - \omega\hat{3} \times \mathbf{S} = \mathbf{\Omega}^{\text{RWA}} \times \mathbf{S} \tag{19.18}$$

where

$$\mathbf{\Omega}^{\mathrm{RWA}} = -R\,\mathbf{e}_1 - \Delta\,\mathbf{e}_3$$

with $\Delta = \omega - \omega_0$ being the laser detuning from atomic transition frequency.

The equations for the components of $\mathbf{S}$ in the rotating frame under RWA can be easily extracted:

$$\frac{d}{dt}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} 0 & \Delta & 0 \\ -\Delta & 0 & R \\ 0 & -R & 0 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} \tag{19.19}$$

It is also instructive to note that the expectation value of the dipole operator

$$\langle\hat{\mathbf{d}}\rangle = \mathbf{d}_{ge}\sigma_{ge} + c.c. = \frac{1}{2}\mathbf{d}_{ge}(S_1 - iS_2) + c.c. = \frac{1}{2}\mathbf{d}_{ge}(u - iv)e^{-i\omega t} + c.c.$$

Hence $(u - iv)$ can be interpreted as the dimensionless part of the dipole moment in the rotating frame.

19.6 RWA Solution

It's easy to see that

$$u^2 + v^2 + w^2 = 1$$

Hence the precessional dynamics of the dipole moment, dipole current and atomic inversion are therefore equivalent to trajectories traced out on the surface of a unit sphere.

For fixed R and Δ, one can easily find the solution to Eq. (19.19). A second derivative of the v equation, followed by substitution from the $\dot{u}$ and $\dot{w}$, leads to

$$\frac{d^2}{dt^2}v + (R^2 + \Delta^2)v = 0$$

with the solution

$$v(t) = v(0)\cos\Omega t + \frac{\dot{v}(0)}{\Omega}\sin\Omega t$$

where $\Omega = \sqrt{R^2 + \Delta^2}$ is the *generalized Rabi frequency*, and is just the length of the vector $\mathbf{\Omega}^{\mathrm{RWA}}$.

With the solution given above for $v(t)$, we can solve for $u(t)$ and $w(t)$ straightforwardly. Remember that $\dot{v}(0) = -\Delta u(0) + Rw(0)$, the results can be expressed as

$$\begin{pmatrix} u(t) \\ v(t) \\ w(t) \end{pmatrix} = \begin{pmatrix} \sin^2\alpha + \cos^2\alpha\,\cos\Omega t & -\cos\alpha\sin\Omega t & \cos\alpha\sin\alpha(\cos\Omega t - 1) \\ \cos\alpha\sin\Omega t & \cos\Omega t & \sin\alpha\sin\Omega t \\ \cos\alpha\sin\alpha(\cos\Omega t - 1) & -\sin\alpha\sin\Omega t & \cos^2\alpha + \sin^2\alpha\,\cos\Omega t \end{pmatrix}\begin{pmatrix} u(0) \\ v(0) \\ w(0) \end{pmatrix}$$

where the angle α is define by

$$\cos\alpha = -\frac{\Delta}{\Omega}, \quad \sin\alpha = \frac{R}{\Omega}$$

For the most commonly encountered case where initially all the atoms are in the ground state, i.e., $[u(0)\ \ v(0)\ \ w(0)] = [0\ \ 0\ \ -1]$, we have

$$\begin{pmatrix} u(t) \\ v(t) \\ w(t) \end{pmatrix} = \begin{pmatrix} \cos\alpha\sin\alpha(1 - \cos\Omega t) \\ -\sin\alpha\sin\Omega t \\ -\cos^2\alpha - \sin^2\alpha\,\cos\Omega t \end{pmatrix}$$

The corresponding solution for the excited state probability $p_e(t)$ is:

$$p_e(t) = \frac{1}{2}[w(t)+1] = \frac{1}{2}\sin^2\alpha(1-\cos\Omega t) = \frac{R^2}{\Omega^2}\,\sin^2\frac{\Omega t}{2}$$

This solution shows the famous "Rabi oscillations". In reality, however, these oscillations are always damped due to relaxation effect.

19.7 Relaxation

So far we have considered only the "coherent evolution", that is, the dynamics of the two-level atom governed by its interaction with the driving field. The resulting dynamics exhibits a simple sinusoidal oscillation — it is reversible. In real situations, this is never the case: due to "relaxations", the system eventually irreversibly relaxes to some steady state.

Relaxation arises from the collection of weak and effectively random perturbations that practically every atomic oscillator is subject to. These perturbations are conventionally attributed to interaction with the "environment" or "reservoir", which means any very large physical system coupled to a single atom in a weak way over a very wide frequency band. The effect of the atom on each reservoir mode is infinitesimal, insignificant for the reservoir, but with cumulative phase memory over the modes and in this way the atom produces a finite back reaction on itself that has a damping effect on its diagonal (population) and off-diagonal (coherence) atomic dynamics. For example, spontaneous emission of an excited atom arises from the interaction between the atom with the vacuum EM modes.

We will study system-reservoir interaction in more detail later. Here we just add the relaxation phenomenologically: adding by hand the decay rates of the population inversion w and the coherence u and v as:

$$\dot{u} = \Delta v - u/T_2 \tag{19.20}$$

$$\dot{v} = -\Delta u + Rw - v/T_2 \tag{19.21}$$

$$\dot{w} = -Rv - (1+w)/T_1 \tag{19.22}$$

The general solutions exhibit a damped oscillatory behavior and are given by

$$\mathcal{X}(t) = Ae^{-at} + (B\cos st + C\sin st)\,e^{-bt} + D$$

where $\mathcal{X}$ stands for either u, v or w. Depending on which one $\mathcal{X}$ denotes, the constant coefficients A, B, C and D take different values, while a, b and s are determined by Δ, R, T_1 and T_2. The steady state is reached when the time derivatives vanish. In the steady state, we have

$$u(\infty) = -\frac{\Delta R T_2^2}{1+(\Delta T_2)^2 + R^2 T_1 T_2}$$

$$v(\infty) = -\frac{R T_2}{1+(\Delta T_2)^2 + R^2 T_1 T_2}$$

$$w(\infty) = -\frac{1+(\Delta T_2)^2}{1+(\Delta T_2)^2 + R^2 T_1 T_2}$$

Without the light field (i.e., $R = 0$), we have

$$(1+w)_t = (1+w)_0\, e^{-t/T_1} \tag{19.23}$$

$$(u-iv)_t = (u-iv)_0\, e^{i\Delta t - t/T_2} \tag{19.24}$$

With relaxation, the length of the vector $u^2+v^2+w^2$ is no longer a constant.

Here T_1 is the decay constant for the population inversion, and T_2 is that for the coherence. These constants are introduced in the study of NMR, where T_1 and T_2 are called the *longitudinal* and *transverse* decay constants, respectively. Let us take a closer look at these two constants. First we realize that things which make the population decay also make the coherence decay. For example, consider two energy levels $|i\rangle$ and $|j\rangle$ and the state vector given by

$$|\psi\rangle = C_i|i\rangle + C_j|j\rangle$$

If the populations in state $|i\rangle$ and $|j\rangle$ decay with rate Γ_i and Γ_j, respectively, i.e., $C_i \sim e^{-\Gamma_i t/2}$ and $C_j \sim e^{-\Gamma_j t/2}$ then

$$\sigma_{ij} = \langle\psi|i\rangle\langle j|\psi\rangle = C_i^* C_j \sim e^{-(\Gamma_i+\Gamma_j)t/2}$$

Second, there exist dephasing mechanisms (elastic scattering) that randomize the phase relationship between C_i and C_j, but leave their magnitude (i.e., population) unchanged. If the dephasing gives rise to a coherence decay rate β_{ij}, then we have

$$\frac{1}{T_2} = \frac{\Gamma_i+\Gamma_j}{2} + \beta_{ij}$$

In a system with dominating dephasing collisions, we have

$$T_2 \ll T_1$$

For the two-level atom we are considering here, if the only relaxation mechanism is the spontaneous emission decay of the excited state with rate Γ, then

$$\frac{1}{T_1} = \Gamma, \quad \frac{1}{T_2} = \frac{\Gamma}{2}$$

19.7.1 Free Induction Decay

In many cases, we need to deal with an ensemble of atoms. The system is usually inhomogeneously broadened due to, e.g., thermal motion of the atoms. The inhomogeneous broadening introduces a distribution of effective detunings. Macroscopic properties of the system are usually averaged over this distribution.

As an example, consider an ensemble two-level atoms with a Lorentzian distribution of frequencies. The laser field is resonant at the peak frequency of the distribution. In the rotating frame, we have the distribution of the effective detuning as

$$P(\Delta) = \frac{\delta\omega}{\pi}\,\frac{1}{\Delta^2+(\delta\omega)^2}$$

where $\delta\omega$ represents the width of the distribution.

Suppose the laser field drives the atoms to the steady state before it is turned off. Then the Bloch vector will evolve according to Eqs. (19.23) and (19.24). In particular the dipole moment evolves as

$$(u - iv)_t = (u - iv)_0 \, e^{i\Delta t - t/T_2}$$

The polarization density associated with these dipoles is

$$\boldsymbol{\mathcal{P}}(t) = \mathcal{N}\mathbf{d}_{eg}\mathrm{Re}\left[\int d\Delta \, P(\Delta)(u - iv)_0 \, e^{i\Delta t - t/T_2} \, e^{-i\omega t}\right]$$

where $\mathcal{N}$ is the atomic density. With the simplifying assumption that the initial values of u_0 and v_0 are Δ-independent, and the Lorentzian distribution, we have

$$\boldsymbol{\mathcal{P}}(t) = \mathcal{N}\mathbf{d}_{eg}\mathrm{Re}\left[(u - iv)_0 \, e^{i\omega t - t/T_2}\right] \, e^{-\delta\omega t}$$

from which we see that the inhomogeneous broadening results in an additional decaying term $\sim e^{-\delta\omega t}$.

Under the condition $\delta\omega \gg 1/T$, this extra term is the dominant decay term. The explanation of the phenomenon, which is called the *free induction decay*, is simple enough. The extra decay factor is due to the interference of all of the dipoles with frequencies distributed throughout the inhomogeneous line. Thus damping due to inhomogeneous broadening may be thought of as a kind of dephasing process that damps only the macroscopic polarization density $\boldsymbol{\mathcal{P}}(t)$. Each individual dipole continues to oscillate for a time T. Well before that time, however, $\boldsymbol{\mathcal{P}}(t)$ may be effectively zero because the dipoles may have drifted completely out of phase with one another.

Chapter 20

Resonant Coupling between a Discrete Level and a Continuum

20.1 Derivation of Fermi's Golden Rule

In many times, we are dealing with the situation where a discrete level $|\varphi_i\rangle$ with energy E_i is coupled to a continuum $|\varphi_f\rangle$ whose bare energy E_f varies continuously. Such examples include:

- *Photoionization* — An atom, initially in a discrete internal state, absorbs a photon having an energy higher than the ionization energy of the atom, and ends up in the state lying in the ionization continuum. This disappearance of a photon, accompanied by the appearance of a photoelectron, is simply the well-known *photoelectric effect*.
- *Photodissociation* — The initial state is a ro-vibrational ground state of a stable molecule. The absorption of a photon brings the molecule to a dissociative excited electronic state and the molecule breaks apart.
- *Spontaneous emission* — Spontaneous emission of an excited atom is another example. Here the final state contains a continuum of photon states.

We split the Hamiltonian into the bare part and the interaction part as

$$H = H_0 + V$$

where the bare or unperturbed Hamiltonian H_0 is assumed to be time independent and $H_0|\varphi_{i,f}\rangle = E_{i,f}|\varphi_{i,f}\rangle$. In the interaction picture, the wave function evolves according to

$$i\hbar\frac{d}{dt}|\psi_I(t)\rangle = V_I(t)\,|\psi_I(t)\rangle$$

where $V_I(t) = U_0^{-1}(t,t_0)VU_0(t,t_0)$ and $U_0(t,t_0) = \exp[-iH_0(t-t_0)/\hbar]$. This can be integrated to give

$$|\psi_I(t)\rangle = |\psi_I(t_0)\rangle + \frac{1}{i\hbar}\int_{t_0}^{t} dt_1\, V_I(t_1)\,|\psi_I(t_1)\rangle$$

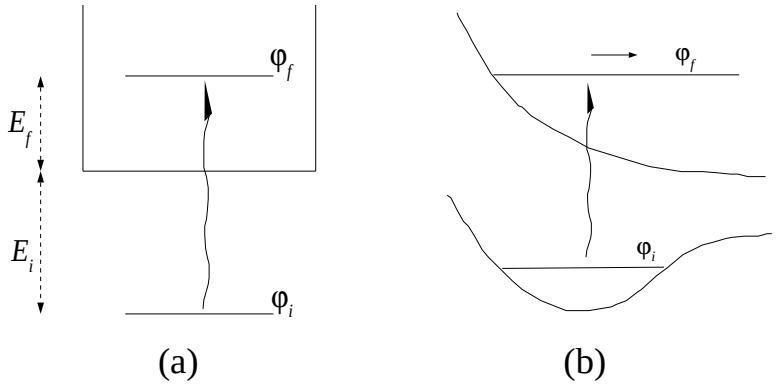

Figure 20-1: (a) Photoionization of an atom; (b) Photodissociation of a molecule.

which is an integral equation that can be solved by iteration. Thus, we may regard $|\psi_I(t_0)\rangle$ as being a zero-order approximation to $|\psi_I(t)\rangle$, and substitute it for $|\psi_I(t_1)\rangle$ in the integrand yields the first-order approximation

$$|\psi_I(t)\rangle = |\psi_I(t_0)\rangle + \frac{1}{i\hbar}\int_{t_0}^{t} dt_1\, V_I(t_1)\,|\psi_I(t_0)\rangle$$

One more iteration we have the second-order result

$$|\psi_I(t)\rangle = |\psi_I(t_0)\rangle + \frac{1}{i\hbar}\int_{t_0}^{t} dt_1\, V_I(t_1)\,|\psi_I(t_0)\rangle + \frac{1}{(i\hbar)^2}\int_{t_0}^{t} dt_1 \int_{t_0}^{t_1} dt_2\, V_I(t_1)V_I(t_2)|\psi_I(t_0)\rangle$$

This procedure can continue indefinitely to yield an infinite series.

For our purpose, assume the interaction is not very strong and/or the interaction time is sufficiently short, such that the first-order result suffices. Now let $|\psi_I(t_0)\rangle = |\varphi_i\rangle$, we want to calculate the transition amplitude to the final state $|\varphi_f\rangle$, assumed to be orthogonal to the initial state, after an interaction time of T. The transition amplitude is given by (take $t_0 = -T/2$ and $t_f = T/2$)

$$A_{fi}(T) = \langle\varphi_f|\psi_I(t_f)\rangle = \frac{1}{i\hbar}\int_{-T/2}^{T/2} dt_1\,\langle\varphi_f|\,V_I(t_1)\,|\varphi_i\rangle = \frac{1}{i\hbar}\int_{-T/2}^{T/2} dt_1\, V_{fi}\, e^{i(E_f-E_i)t_1/\hbar}$$

where $V_{fi} = \langle\varphi_f|\,V\,|\varphi_i\rangle$ has assumed to be time-independent. Then we have

$$A_{fi}(T) = -2\pi i V_{fi}\,\delta^{(T)}(E_f - E_i)$$

where

$$\delta^{(T)}(E_f - E_i) = \frac{1}{2\pi\hbar}\int_{-T/2}^{T/2} dt_1\, e^{i(E_f-E_i)t_1/\hbar} = \frac{1}{\pi}\frac{\sin[(E_f-E_i)T/(2\hbar)]}{E_f - E_i}$$

is the diffraction function. This function peaks at $E_f - E_i = 0$ with the maximum value $T/(2\pi\hbar)$, and its width is on the order of $4\pi\hbar/T$ (distance between the first two zeros on either side of the peak). Its integral over E_f equals one. All these properties show that $\delta^{(T)}(E_f - E_i)$ behaves like a delta function. In fact, in the limit $T \to \infty$, $\delta^{(T)}(E_f - E_i) \to \delta(E_f - E_i)$.

The transition probability is given by $|A_{fi}(T)|^2$, for which we need to evaluate

$$[\delta^{(T)}(E_f - E_i)]^2 = \frac{1}{\pi^2}\frac{\sin^2[(E_f-E_i)T/(2\hbar)]}{(E_f - E_i)^2}$$

which peaks at $E_f - E_i = 0$ with maximum value $T^2/(2\pi\hbar)^2$, and its width is again on the order of $4\pi\hbar/T$. In addition, we can work out the integral over E_f as

$$\begin{aligned}\int_{-\infty}^{\infty} dE_f\,[\delta^{(T)}(E_f - E_i)]^2 &= \frac{1}{(2\pi\hbar)^2}\int_{-\infty}^{\infty} dE_f \int_{-T/2}^{T/2} dt_1\, e^{i(E_f-E_i)t_1/\hbar} \int_{-T/2}^{T/2} dt_2\, e^{i(E_f-E_i)t_2/\hbar} \\ &= \frac{1}{(2\pi\hbar)^2}\int_{-\infty}^{\infty} dE_f\, e^{iE_f(t_1+t_2)/\hbar} \int_{-T/2}^{T/2} dt_1 \int_{-T/2}^{T/2} dt_2 e^{-iE_i(t_1+t_2)/\hbar} \\ &= \frac{1}{(2\pi\hbar)^2}\int_{-T/2}^{T/2} dt_1 \int_{-T/2}^{T/2} dt_2\, 2\pi\hbar\delta(t_1+t_2) e^{-iE_i(t_1+t_2)/\hbar} \\ &= \frac{T}{2\pi\hbar}\end{aligned}$$

Therefore we are justified to write

$$[\delta^{(T)}(E_f - E_i)]^2 \simeq \frac{T}{2\pi\hbar}\delta^{(T)}(E_f - E_i)$$

Therefore the transition probability from $|\varphi_i\rangle$ to $|\varphi_f\rangle$ is proportional to the duration of the interaction T, which allows us to define a transition rate equal to

$$w_{fi} = \frac{1}{T}|A_{fi}(T)|^2 \simeq \frac{2\pi}{\hbar}|V_{fi}|^2\delta^{(T)}(E_f - E_i)$$

The final state $|\varphi_f\rangle$, which belongs to a continuum, is not normalizable. The quantity that does have a physical meaning is the transition rate toward a group of final states. For example, the sum of the above equation over all the states $|\varphi_f\rangle$ gives the transition rate Γ of the discrete state $|\varphi_i\rangle$ to any state of the continuum:

$$\Gamma = \sum_f w_{fi} = \frac{2\pi}{\hbar}\sum_f |V_{fi}|^2\delta^{(T)}(E_f - E_i) = \frac{2\pi}{\hbar}|V_{fi}|^2\rho(E_f = E_i)$$

where $|V_{fi}|$ is assumed to be dependent on E_f only and $\rho(E_f = E_i)$ is the density of final states evaluated at $E_f = E_i$. This is just *Fermi's Golden Rule* which is very useful over a variety of problems.

20.2 Semiclassical Theory of Photoelectric Effect

We can straightforwardly apply Fermi's Golden Rule to explain the photoelectric (photoionization) effect. Here the initial state is an atom in a bound state with energy $E_0 < 0$ (we take the bottom of the continuum as the energy reference level) and a photon with frequency ω, hence $E_i = E_0 + \hbar\omega$. The coupling between the bound state and the continuum can be treated using the dipole approximation: $V = \hat{\mathbf{d}} \cdot \mathcal{E}_0\boldsymbol{\epsilon}$ where $\mathcal{E}_0$ and $\boldsymbol{\epsilon}$ are the amplitude and the unit polarization vector of the light field, respectively. Here we use the semiclassical theory where the light field is treated classically. Then the transition rate is given by

$$\Gamma = \frac{2\pi}{\hbar}|\mathbf{d}_{fi} \cdot \boldsymbol{\epsilon}|^2\mathcal{E}_0^2\rho(E_0 + \hbar\omega)$$

where $\mathbf{d}_{fi} = \langle\varphi_f|\hat{\mathbf{d}}|\varphi_i\rangle$ is the dipole transition matrix element.

Since all the final states lie above $E = 0$, hence $\rho(E) = 0$ if $E < 0$, which implies a minimum threshold photon frequency $\hbar\omega \geq |E_0|$ for photoemission, and as long as this condition is satisfied, there is a non-vanishing probability for photoelectron emission to occur as soon as the detector is exposed to the light, no

matter how weak the field may be. This shows that the particle concept of photon is not required to explain the photoelectric effect, contrary to conventional wisdom.

Remark: The preceding discussion seems to indicate that the probability of photoionization is zero for $\hbar\omega < |E_0|$ because the absorption of a photon cannot bring the atom into the ionization continuum. This is correct for weak light intensity for which the above perturbative treatment is valid. At high intensity, the atom can be photoionized even if $\hbar\omega < |E_0|$. This occurs through a multiphoton ionization process, during which the atom absorbs n photons, bringing to it an energy $n\hbar\omega$ sufficient to pass into the ionization continuum.

Chapter 21

Multi-Photon Resonant Coupling

So far we have dealt with the processes of emission and absorption that involve only a single photon. More complex processes may occur in which the number of photons may increase or decrease by several units. Such processes are called *multi-photon* processes. There are many types of multi-photon processes. Here we will discuss one particular example as illustrated in Fig. 1 which describes a two-photon process: State $|0\rangle$ is connected to $|2\rangle$ by a two-photon coupling via a set of intermediate states $|\{j\}\rangle$.

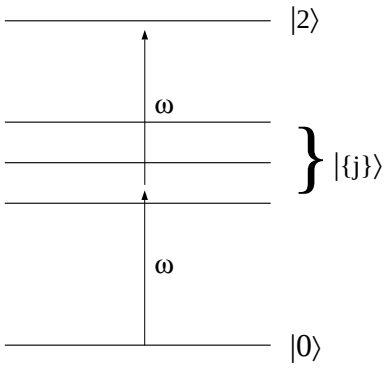

Figure 21-1: A resonant two-photon coupling scheme.

Take the bare energy of state $|0\rangle$ to be the energy reference. States $|2\rangle$ and $|\{j\}\rangle$ have energies $\hbar\omega_2$ and $\hbar\omega_j$, respectively. The frequency of the light is ω. We consider the situation that the two-photon coupling between $|0\rangle$ and $|2\rangle$ is near resonant, i.e., the two-photon detuning

$$\Delta^{(2)} = 2\omega - \omega_2 \approx 0$$

but none of the one-photon coupling between $|0\rangle$ and $|\{j\}\rangle$, or between $|\{j\}\rangle$ and $|2\rangle$ are resonant.

The Hamiltonian under the RWA can be written as

$$H = \sum_j \hbar\omega_j |j\rangle\langle j| + \hbar\omega_2 |2\rangle\langle 2| + \frac{\hbar}{2}\sum_j \left(\Omega_{j0} e^{-i\omega t} |j\rangle\langle 0| + \Omega_{j0}^* e^{i\omega t} |0\rangle\langle j|\right) + \frac{\hbar}{2}\sum_j \left(\Omega_{2j} e^{-i\omega t} |2\rangle\langle j| + \Omega_{2j}^* e^{i\omega t} |j\rangle\langle 2|\right)$$

The atomic wave function has the following general form

$$|\psi(t)\rangle = c_0(t)\,|0\rangle + \sum_j c_j(t)\,e^{-i\omega_j t}\,|j\rangle + c_2(t)\,e^{-i\omega_2 t}\,|2\rangle$$

Initially the atom is in state $|0\rangle$, therefore

$$c_0(0) = 1, \quad c_j(0) = c_2(0) = 0$$

With these expression, we have

$$i\hbar\frac{\partial}{\partial t}|\psi(t)\rangle = i\hbar\left[\dot{c}_0|0\rangle + \sum_j \left(\dot{c}_j e^{-i\omega_j t} - i\omega_j c_j e^{-i\omega_j t}\right)|j\rangle + \left(\dot{c}_2 e^{-i\omega_2 t} - i\omega_2 c_2 e^{-i\omega_2 t}\right)|2\rangle\right]$$

and

$$\begin{aligned} H|\psi(t)\rangle &= \sum_j \hbar\omega_j c_j e^{-i\omega_j t}|j\rangle + \hbar\omega_2 c_2 e^{-i\omega_2 t}|2\rangle + \frac{\hbar}{2}\sum_j \Omega_{j0}e^{-i\omega t}c_0|j\rangle + \frac{\hbar}{2}\sum_j \Omega^*_{j0}e^{i(\omega-\omega_j)t}c_j|0\rangle \\ &\quad + \frac{\hbar}{2}\sum_j \Omega_{2j}e^{-i(\omega+\omega_j)t}c_j|2\rangle + \frac{\hbar}{2}\sum_j \Omega^*_{2j}e^{i(\omega-\omega_2)t}c_2|j\rangle \end{aligned}$$

Equating these two equations we obtain the equations of motion for the amplitudes:

$$i\dot{c}_0 = \frac{1}{2}\sum_j \Omega^*_{j0}\,e^{i(\omega-\omega_j)t}\,c_j \tag{21.1}$$

$$i\dot{c}_j = \frac{1}{2}\Omega_{j0}\,e^{-i(\omega-\omega_j)t}\,c_0 + \frac{1}{2}\Omega^*_{2j}\,e^{i(\omega-\omega_2+\omega_j)t}\,c_2 \tag{21.2}$$

$$i\dot{c}_2 = \frac{1}{2}\sum_j \Omega_{2j}\,e^{-i(\omega-\omega_2+\omega_j)t}\,c_j \tag{21.3}$$

Equation (21.2) can be formally integrated as

$$ic_j(t) = \frac{1}{2}\Omega_{j0}\int_0^t dt'\,e^{-i(\omega-\omega_j)t'}\,c_0(t') + \frac{1}{2}\Omega^*_{2j}\int_0^t dt'\,e^{i(\omega-\omega_2+\omega_j)t'}\,c_2(t') \tag{21.4}$$

Let us consider the first integral at the r.h.s. Integrate by parts we have

$$\begin{aligned} \int_0^t dt'\,e^{-i(\omega-\omega_j)t'}\,c_0(t') &= \frac{i}{\omega-\omega_j}\,e^{-i(\omega-\omega_j)t'}\,c_0(t')\Big|_0^t - \frac{i}{\omega-\omega_j}\int_0^t dt'\,e^{-i(\omega-\omega_j)t'}\,\dot{c}_0(t') \\ &= \frac{i}{\omega-\omega_j}\left[e^{-i(\omega-\omega_j)t}\,c_0(t) - 1\right] - \frac{i}{\omega-\omega_j}\int_0^t dt'\,e^{-i(\omega-\omega_j)t'}\,\dot{c}_0(t') \end{aligned} \tag{21.5}$$

Since the transition between $|0\rangle$ and $|\{j\}\rangle$ are off-resonant, we expect that c_j is small, hence according to Eq. (21.1), $\dot{c}_0$ is small. Therefore we neglect the last term at the r.h.s. of (21.5).

Performing the similar procedure to the second integral of (21.4), we have

$$c_j(t) = \frac{\Omega_{j0}}{2}\frac{e^{-i(\omega-\omega_j)t}\,c_0(t) - 1}{\omega-\omega_j} - \frac{\Omega^*_{2j}}{2}\frac{e^{i(\omega-\omega_2+\omega_j)t}\,c_2(t)}{\omega-\omega_2+\omega_j}$$

Putting back into Eqs. (21.1) and (21.3), we have

$$i\dot{c}_0 = \frac{1}{4}\sum_j \frac{|\Omega_{j0}|^2}{\omega-\omega_j}\left[c_0 - e^{i(\omega-\omega_j)t}\right] - \frac{1}{4}\sum_j \frac{\Omega^*_{j0}\Omega^*_{2j}}{\omega-\omega_2+\omega_j}\,e^{i\Delta^{(2)}t}\,c_2 \tag{21.6}$$

$$i\dot{c}_2 = \frac{1}{4}\sum_j \frac{\Omega_{2j}\Omega_{j0}}{\omega-\omega_j}\left[e^{-i\Delta^{(2)}t}\,c_0 - e^{-i(\omega-\omega_2+\omega_j)t}\right] - \frac{1}{4}\sum_j \frac{|\Omega_{2j}|^2}{\omega-\omega_2+\omega_j}\,c_2 \tag{21.7}$$

Under the assumption of off-resonant one-photon coupling, the second terms inside the square bracket represent fast-oscillating terms, which can be neglected. Furthermore, we can define

$$\delta_0 = -\frac{1}{4}\sum_j \frac{|\Omega_{j0}|^2}{\omega-\omega_j}, \quad \delta_2 = \frac{1}{4}\sum_j \frac{|\Omega_{2j}|^2}{\omega-\omega_2+\omega_j}, \quad \Omega^{(2)} = \frac{1}{2}\sum_j \frac{\Omega_{2j}\Omega_{j0}}{\omega-\omega_j} \approx -\frac{1}{2}\sum_j \frac{\Omega_{2j}\Omega_{j0}}{\omega-\omega_2+\omega_j}$$

In the last equation, we have used $2\omega-\omega_2 \approx 0$, hence $\omega-\omega_j \approx -(\omega-\omega_2+\omega_j)$. With these definitions, and neglect the non-resonant terms, Eqs. (21.6) and (21.7) can be rewritten as

$$i\dot{c}_0 = -\delta_0 c_0 + \frac{1}{2}\left[\Omega^{(2)}\right]^* e^{i\Delta^{(2)}t}\, c_2 \tag{21.8}$$

$$i\dot{c}_2 = -\delta_2 c_2 + \frac{1}{2}\Omega^{(2)} e^{-i\Delta^{(2)}t}\, c_0 \tag{21.9}$$

which describes a dipole coupling between an effective two-level atom consisting states $|0\rangle$ and $|2\rangle$. $\Omega^{(2)}$ can be identified as the effective two-photon Rabi frequency. The two levels of this effective two-level atom obtain additional shifts δ_0 and δ_2, respectively. These are called the *AC Stark shifts*.

The intermediate states $|\{j\}\rangle$ have all been eliminated. This procedure is called *adiabatic elimination* which is valid when these states are not resonantly coupled.

Chapter 22

Photon Echo

We have introduced the concept of free induction decay earlier. In an inhomogeneously broadened medium, macroscopic polarization density $\boldsymbol{\mathcal{P}}$ quickly decays away long before time T_2 (the time for individual atomic dipole moment to decay) due to dephasing of individual dipoles. Although irreversibility and decay are powerful partners that dominate much of many-body physics, some decay phenomena are actually reversible. Free induction decay is one of them. The reverse of the free induction decay can be demonstrated beautifully with *photon echo*.

22.1 Qualitative Consideration

Consider an ensemble of Bloch vectors initially located at $(0, 0, -1)$. At $t = 0$, an intense $\pi/2$ near-resonant short pulse will rotate them to the $\mathbf{e}_2$-axis. After the pulse, the vectors will experience free evolution, i.e., a precession along the $\mathbf{e}_3$-axis. Suppose they have different detunings, hence they precess at different frequencies, and quickly get out of phase (i.e., free induction decay). Now we want to see whether it is possible to convince these out-of-phase Block vectors to become in-phase again via a reversal of the initial dephasing.

Before discuss further we can take a analogous diffusion with reversal followed by realignment in a different field as shown in the figure. At $t = 0$, the race starts. Runners quickly "diffuse" as they have different speed. At $t = \Delta t$, however, someone fires a gun, signalling all the runners to reverse their direction. Then one can expect at $t = 2\Delta t$, all the runners will get back to the starting line again, i.e., rephasing has occurred.

Armed with this important insights, we can try to see if we can similar things to the Bloch vectors. It is however impossible to realign the dephased Bloch vectors in the same way as the runners, because the direction of turning of each vector is fixed by its detuning. Hence one can not reverse the direction of its free precession. The Bloch vectors however have a freedom that the runners do not: they are not confined to the horizontal plane. It is easy to see that a rotation of all the vectors about the $\mathbf{e}_1$-axis through $180°$, which can be achieved via a short π-pulse, produces a collection of rephasing moments. The reformed macroscopic moment is in the opposite direction of the original one.

22.2 Quantitative Consideration

Now we want to treat the problem more quantitatively. The pulse sequence is shown in the figure below.

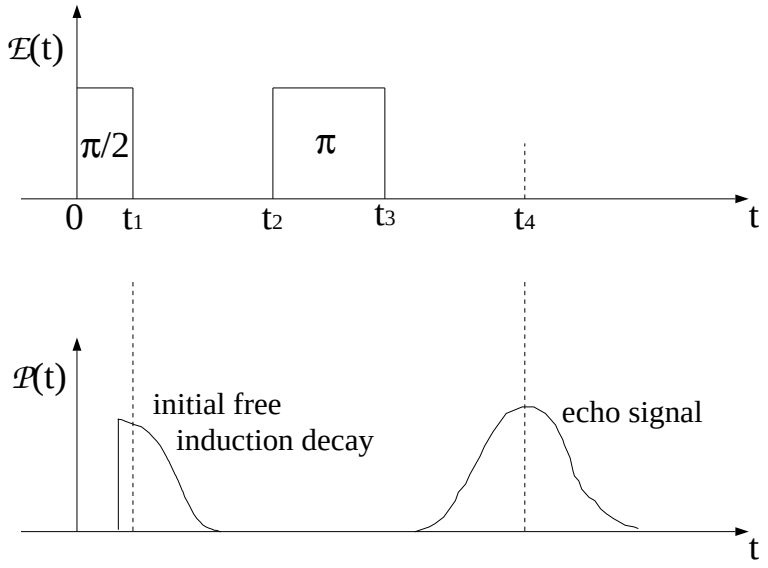

Figure 22-1: Photon echo time sequence.

The time scale we are interested in here is much shorter than T_1 and T_2, but longer than the inhomogeneous decay time $1/\delta\omega$. Hence in the OBEs for each individual atom, we can neglect the decay terms. Without these decay terms, the solution to the OBEs can be cast into the matrix form as

$$\begin{pmatrix} u(t) \\ v(t) \\ w(t) \end{pmatrix} = \begin{pmatrix} \frac{R^2}{\Omega^2} + \frac{\Delta^2}{\Omega^2}\cos\Omega t & \frac{\Delta}{\Omega}\sin\Omega t & -\frac{R\Delta}{\Omega^2}(\cos\Omega t - 1) \\ -\frac{\Delta}{\Omega}\sin\Omega t & \cos\Omega t & \frac{R}{\Omega}\sin\Omega t \\ -\frac{R\Delta}{\Omega^2}(\cos\Omega t - 1) & -\frac{R}{\Omega}\sin\Omega t & \frac{\Delta^2}{\Omega^2} + \frac{R^2}{\Omega^2}\cos\Omega t \end{pmatrix} \begin{pmatrix} u(0) \\ v(0) \\ w(0) \end{pmatrix}$$

At $t = 0$, $[u(0)\ v(0)\ w(0)] = [0\ 0\ -1]$. We can solve the dynamics by finding appropriate matrices. At $0 < t < t_1$, a $\pi/2$ pulse is applied, i.e., $\Omega t_1 = \pi/2$. Furthermore, we assume that the light field is very intense such that $R \gg |\Delta|$, hence we neglect terms to the second or higher orders of Δ/R. The effect of this first pulse at the end of $t = t_1$ can be characterized by the matrix

$$\mathbf{M}_1 = \begin{pmatrix} 1 & \Delta/R & \Delta/R \\ -\Delta/R & 0 & 1 \\ \Delta/R & -1 & 0 \end{pmatrix}$$

At $t_1 < t < t_2$, the system goes through a free evolution, whose effect can be captured by the matrix

$$\mathbf{M}_2 = \begin{pmatrix} \cos\Delta t_{21} & \sin\Delta t_{21} & 0 \\ -\sin\Delta t_{21} & \cos\Delta t_{21} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with $t_{21} = t_2 - t_1$. At $t_2 < t < t_3$, a π pulse is applied, i.e., $\Omega t_{32} = \pi$. Again we assume $R \gg \Delta$, hence

$$\mathbf{M}_3 = \begin{pmatrix} 1 & 0 & 2\Delta/R \\ 0 & -1 & 0 \\ 2\Delta/R & 0 & -1 \end{pmatrix}$$

From t_3 to t_4 we again have a free evolution with

$$\mathbf{M}_4 = \begin{pmatrix} \cos\Delta t_{43} & \sin\Delta t_{43} & 0 \\ -\sin\Delta t_{43} & \cos\Delta t_{43} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The Bloch vector at $t = t_4$ can then be straightforwardly calculated as

$$\begin{pmatrix} u(t_4) \\ v(t_4) \\ w(t_4) \end{pmatrix} = \mathbf{M}_4\,\mathbf{M}_3\,\mathbf{M}_2\,\mathbf{M}_1 \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} = \begin{pmatrix} \sin\Delta(t_{43}-t_{21}) - \frac{\Delta}{R}\cos\Delta(t_{43}-t_{21}) \\ \cos\Delta(t_{43}-t_{21}) + \frac{\Delta}{R}\sin\Delta(t_{43}-t_{21}) \\ -2\frac{\Delta}{R}\sin\Delta t_{21} \end{pmatrix}$$

from which we have

$$\begin{aligned} u(t_4;\Delta) - iv(t_4;\Delta) &= -\left(i + \frac{\Delta}{R}\right) e^{i\Delta(t_{43}-t_{21})} \\ &\approx -ie^{i\Delta(t_{43}-t_{21}-1/R)} \end{aligned}$$

Therefore when $t_{43} - t_{21} - 1/R = 0$, i.e., at $t_4 = t_3 + t_{21} + 1/R$, regardless of the value of Δ, the dimensionless dipole moment

$$u - iv \approx -i$$

as depicted in the following figure. Thus a macroscopic polarization density will result at time $t = t_4$.

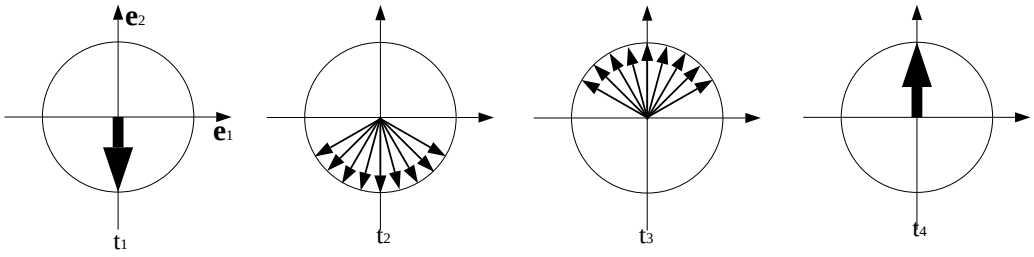

Figure 22-2: Evolution of the dipole moment.

The echo phenomenon was first discovered by Hahn in nuclear spin in 1950. Photon echoes were first observed in 1964 by Kurnit, Abella and Hartmann.

Chapter 23

Pulsed Excitation

So far we have taken the electric field to have a constant amplitude. In many situations, we are dealing with pulsed excitations. Consider a field given by

$$\mathbf{E}(t) = \mathcal{E}(t)\boldsymbol{\epsilon}\left(e^{i\omega t} + e^{-i\omega t}\right)$$

where we now allow the amplitude $\mathcal{E}(t)$ to be time-dependent. We assume the pulse is sufficiently short that the time of interest is much shorter than the relaxation time.

23.1 Amplitude Equations

The field is interacting with a two-level atom with a state vector given by

$$|\psi(t)\rangle = c_g|g\rangle + c_e(t)e^{-i\omega_0 t}|e\rangle$$

and the interaction Hamiltonian is given by

$$H_{\text{int}} = -\mathbf{d}\cdot\boldsymbol{\epsilon}\mathcal{E}(t)\left(e^{i\omega t}|g\rangle\langle e| + e^{-i\omega t}|e\rangle\langle g|\right)$$

From these we have

$$\dot{c}_g(t) = i\frac{1}{2}R(t)c_e(t) \tag{23.1}$$

$$\dot{c}_e(t) = i\frac{1}{2}R(t)c_g(t) \tag{23.2}$$

where $R(t) = 2\mathbf{d}\cdot\boldsymbol{\epsilon}\mathcal{E}(t)/\hbar$ is the time-dependent Rabi frequency and we have assumed, for simplicity, $\omega = \omega_0$ (i.e., exact resonance). Consider initially the atom is in its ground state, i.e.,

$$c_g(0) = 1, \quad c_e(0) = 0$$

From Eqs. (23.1) and (23.2), we have

$$\dot{c}_g(t)c_g(t) = \dot{c}_e(t)c_e(t)$$

Now define $c'_e = -ic_e$, we have

$$\dot{c}_g(t)c_g(t) + \dot{c}'_e(t)c'_e(t) = \frac{d}{dt}\left(c_g^2 + (c'_e)^2\right) = 0$$

i.e., the quantity $c_g^2 + (c'_e)^2$ is a constant of motion. It's easy to see from the initial condition that

$$c_g^2 + (c'_e)^2 = 1$$

Hence we can define

$$c_g(t) = \cos\frac{\Theta(t)}{2}, \quad c'_e(t) = -ic_e(t) = \sin\frac{\Theta(t)}{2}$$

Using these definitions, Eq. (23.1) becomes

$$\dot{c}_g(t) = -\frac{1}{2}\dot{\Theta}(t)\sin\frac{\Theta(t)}{2} = -\frac{1}{2}R\sin\frac{\Theta(t)}{2}$$

Therefore we have

$$\dot{\Theta}(t) = R(t), \text{ or } \Theta(t) = \int_0^t R(t')\,dt'$$

Θ is known as the *pulse area*. A π-pulse corresponds to $\Theta(t \to \infty) = \pi$, after which the population is completely inverted from the ground state to the excited state.

In realistic situations with multiple atomic levels, the bandwidth of the pulse becomes an issue. The bandwidth, $\Delta\omega$, of the pulse is roughly the inverse of the pulse duration $\Delta\tau$. For femto-second pulse, $\Delta\omega \approx 10^{15}$Hz, which is enough to cover all the levels of the hydrogen atom with principle quantum number $n \geq 2$.

23.2 Bloch Vector Behavior and finite detuning

We can restudy this problem in terms of BLoch vector. The equations of motion for Bloch vector are

$$\dot{u} = \Delta v \tag{23.3}$$

$$\dot{v} = -\Delta u + Rw \tag{23.4}$$

$$\dot{w} = -Rv \tag{23.5}$$

At resonance ($\Delta = 0$), it is easy to see that u is now a constant of motion and v and w can be solved as

$$v(t;0) = -\sin\Theta(t), \quad w(t;0) = -\cos\Theta(t)$$

where '0' in the argument of v and w represents $\Delta = 0$.

To generalize this to finite detuning is not straightforward. Due to the time-dependence of $R = \dot{\Theta}$, the Rabi oscillation solution cannot be expected to be relevant here. However, it may be reasonable to expect that the off-resonance dipoles respond to the pulse in the same way as the resonant dipoles, but with a detuning-dependent reduction in amplitude. Thus we assume the validity of the following ansatz:

$$v(t;\Delta) = v(t;0)F(\Delta) = -F(\Delta)\,\sin\Theta(t) \tag{23.6}$$

where the dimensionless $F(\Delta)$ is called the diple *spectral response* function.

With such an ansatz, Eq. (25.3) can be immediately solved to give

$$w(t;\Delta) = -1 + F(\Delta)\,[1 - \cos\Theta(t)]$$

Eq. (24.20) yields

$$\Delta\dot{u} = -\ddot{v} + \dot{R}w + R\dot{w} = [F(\Delta) - 1]\,\ddot{\Theta}(t) \tag{23.7}$$

On the other hand, according to Eq. (23.3), we have

$$\Delta\dot{u} = \Delta^2 v = -\Delta^2 F(\Delta)\,\sin\Theta(t) \tag{23.8}$$

Equating (23.7) and (23.8), we obtain an equation for Θ only:

$$\ddot{\Theta}(t) - \frac{1}{\tau_p^2}\,\sin\Theta(t) = 0 \tag{23.9}$$

where

$$\frac{1}{\tau_p^2} = \frac{\Delta^2 F(\Delta)}{1 - F(\Delta)}, \quad \text{or} \quad F(\Delta) = \frac{1}{1 + (\Delta\tau_p)^2}$$

Note that since neither $\ddot{\Theta}$ nor $\sin\Theta$ depend on Δ, the coefficient τ_p cannot depend on Δ either.

Equation (23.9) is known as the *pendulum equation* familiar from the theory of the pendulum. It has the full range of elliptic function solutions typical of pendulum problem. Only one of these solutions is appropriate to our case, i.e., the one that fit the boundary condition that both $\mathcal{E} \sim \dot{\Theta}$ and $\dot{\mathcal{E}} \sim \ddot{\Theta}$ must vanish at $t = \pm\infty$. Such restrictions imply a non-oscillating pendulum, a situation that is possible only if the pendulum was balanced vertically in the infinite past and then falls and swings once completely over to end up just balanced vertically again in the infinite future. The associated solution is

$$\Theta(t) = 4\tan^{-1}\left(e^{(t-t_0)/\tau_p}\right)$$

or translate to the electric field envelope as

$$\mathcal{E}(t) = \frac{2}{\kappa\tau_p}\,\mathrm{sech}\left(\frac{t - t_0}{\tau_p}\right) \tag{23.10}$$

with $\kappa \equiv 2\mathbf{d}\cdot\boldsymbol{\epsilon}/\hbar$. Eq. (23.10) represents the famous hyperbolic secant pulse found by McCall and Hahn in 1967, from which we can identify τ_p as the pulse width. Obviously, this is a 2π pulse.

The corresponding solution for the Bloch vector is

$$\begin{aligned}
u &= -\frac{2\Delta\tau_p}{1 + (\Delta\tau_p)^2}\,\mathrm{sech}\left(\frac{t - t_0}{\tau_p}\right) \\
v &= \frac{2}{1 + (\Delta\tau_p)^2}\,\mathrm{sech}\left(\frac{t - t_0}{\tau_p}\right)\tanh\left(\frac{t - t_0}{\tau_p}\right) \\
w &= -1 + \frac{2}{1 + (\Delta\tau_p)^2}\,\mathrm{sech}^2\left(\frac{t - t_0}{\tau_p}\right)
\end{aligned}$$

from which it is easy to verify that the conservation law $u^2 + v^2 + w^2 = 1$ is exactly fulfilled.

Hyperbolic secant pulse has fascinating properties. Later we will revisit these pulses from the perspective of the Maxwell-Bloch equations. The fact that we are able to derive these hyperbolic secant pulse solutions is quite remarkable, since we haven't incorporated the Maxwell's equations — the equations governing the dynamics of the field — anywhere in our formulation. This shows the strength of the constraints that the nonlinearities of the quantum dipole equations impose on the phenomena of resonance optics.

Chapter 24

Maxwell-Bloch Equations

So far we have focused on atoms as they respond to near resonant radiation fields. Now we will pay attention to the reverse process, the response of the field to the atoms. Such response must exist since the polarization density (e.g., arising from atomic dipoles) enters Maxwell's equations as a source term. Whenever the sourse polarization is affected by the field that it generates, the issue of self-consistency will arise.

24.1 Field Equations and Dipole Sources

In non-magnetic media with no free charges, the electric field $\mathbf{E}$ and polarization density $\mathbf{P}$ are connected by the Maxwell's wave equation:

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\mathbf{E}(\mathbf{r},t) = \frac{1}{\varepsilon_0 c^2}\frac{\partial^2}{\partial t^2}\mathbf{P}(\mathbf{r},t) \tag{24.1}$$

For simplicity, let us consider the situation the field is linearly polarized, say along x-axis, plane wave propagating along z-axis. Hence we neglect the vector nature of $\mathbf{E}$ and $\mathbf{P}$, and their magnitude depend only on z. Hence the above equation is reduced to

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)E(z,t) = \frac{1}{\varepsilon_0 c^2}\frac{\partial^2}{\partial t^2}P(z,t) \tag{24.2}$$

We have already found that the time-dependent atomic dynamics is easier to treat in the rotating frame since the Block vector moves much slower in this rotating frame. Now we will follow a similar procedure in treating evolution of the electric field. To this end, we separate off a plane-wave carrier factor, which oscillates very rapidly in both space and time

$$E(z,t) = \tilde{\mathcal{E}}(z,t)\,e^{-i(\omega t - kz)} + c.c., \quad P(z,t) = \tilde{\mathcal{P}}(z,t)\,e^{-i(\omega t - kz)} + c.c.$$

The envelope functions $\tilde{\mathcal{E}}(z,t)$ and $\tilde{\mathcal{P}}(z,t)$ still depend on both z and t, but its spatio-temporal dependence is much slower. Quantitatively, this means

$$k^2|\tilde{\mathcal{E}}(z,t)| \gg k\left|\frac{\partial\tilde{\mathcal{E}}(z,t)}{\partial z}\right| \gg \left|\frac{\partial^2\tilde{\mathcal{E}}(z,t)}{\partial z^2}\right|, \quad \omega^2|\tilde{\mathcal{E}}(z,t)| \gg \omega\left|\frac{\partial\tilde{\mathcal{E}}(z,t)}{\partial t}\right| \gg \left|\frac{\partial^2\tilde{\mathcal{E}}(z,t)}{\partial t^2}\right|$$

and similarly for $\tilde{\mathcal{P}}(z,t)$. This is called the *slowly-varying envelope approximation* (SVEA).

Using the SVEA, we have

$$\left(\frac{\partial^2}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)E = \left[\frac{\partial^2\tilde{\mathcal{E}}}{\partial z^2}+2ik\frac{\partial\tilde{\mathcal{E}}}{\partial z}-k^2\tilde{\mathcal{E}}\right]e^{-i(\omega t-kz)}-\left[\frac{1}{c^2}\frac{\partial^2\tilde{\mathcal{E}}}{\partial t^2}-2i\frac{\omega}{c^2}\frac{\partial\tilde{\mathcal{E}}}{\partial z}-\frac{\omega^2}{c^2}\tilde{\mathcal{E}}\right]e^{-i(\omega t-kz)}+c.c.$$
$$\approx 2ik\left(\frac{\partial\tilde{\mathcal{E}}}{\partial z}+\frac{1}{c}\frac{\partial\tilde{\mathcal{E}}}{\partial t}\right)e^{-i(\omega t-kz)}+c.c.$$

and similarly

$$\frac{\partial^2}{\partial t^2}P=\left[\frac{\partial^2\tilde{\mathcal{P}}}{\partial t^2}-2i\omega\frac{\partial\tilde{\mathcal{P}}}{\partial z}-\omega^2\tilde{\mathcal{P}}\right]e^{-i(\omega t-kz)}+c.c.\approx-\omega^2\tilde{\mathcal{P}}\,e^{-i(\omega t-kz)}+c.c.$$

Therefore the SVEA form of Eq. (24.2) becomes

$$\left(\frac{\partial}{\partial z}+\frac{1}{c}\frac{\partial}{\partial t}\right)\tilde{\mathcal{E}}=\frac{ik}{2\varepsilon_0}\tilde{\mathcal{P}} \tag{24.3}$$

Now let us decompose the field envelope into its real magnitude and phase as

$$\tilde{\mathcal{E}}(z,t)\equiv\mathcal{E}(z,t)\,e^{-i\phi(z,t)} \tag{24.4}$$

Note that a time-dependent ϕ implies a frequency that shifts with time (known as the frequency chirp), and introduces the notion of an instantaneous frequency

$$\omega(t)\equiv\omega+\dot{\phi}(z,t)$$

When we come to the rotating frame transformation, we will use the instantaneous frequency, which requires a modification of the rotating wave transformation accordingly as

$$\begin{aligned}\dot{u} &= (\Delta+\dot{\phi})\,v\\ \dot{v} &= -(\Delta+\dot{\phi})\,u+Rw\\ \dot{w} &= -Rv\end{aligned}$$

The transverse components of the Bloch vector in the lab frame and those in the rotating frame are related as

$$S_1-iS_2\rightarrow(u-iv)e^{-i[\omega t+\phi(z,t)]}$$

and the polarization density is then

$$\tilde{\mathcal{P}}(z,t)\equiv\frac{1}{2}\mathcal{N}d(u-iv)e^{-i\phi(z,t)} \tag{24.5}$$

with $\mathcal{N}$ being the atomic density.

With (24.4) and (24.5), we can separate Eq. (24.3) into two real equations for $\mathcal{E}$ and ϕ as

$$\left(\frac{\partial}{\partial z}+\frac{1}{c}\frac{\partial}{\partial t}\right)\mathcal{E} = \frac{\pi\mathcal{N}d\omega}{4\varepsilon_0 c}\,v \tag{24.6}$$

$$\mathcal{E}\left(\frac{\partial}{\partial z}+\frac{1}{c}\frac{\partial}{\partial t}\right)\phi = -\frac{\pi\mathcal{N}d\omega}{4\varepsilon_0 c}\,u \tag{24.7}$$

Therefore we see that v is the absorptive part of the dipole moment, responsible for changes in the field amplitude ($\mathcal{E}$), while u is the dispersive part, giving rise to changes in the field phase (ϕ).

Since in the Bloch equations, the effects of the incident field $\mathcal{E}$ on the atom are expressed in terms of the Rabi frequency $R = 2d\mathcal{E}/\hbar$, it is natural then to convert the propagation equation for $\mathcal{E}$ in to a propagation equation for R. From (24.6) and (24.7), we have

$$\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right) R = \frac{\mu}{2} v \tag{24.8}$$

$$R\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right) \phi = -\frac{\mu}{2} u \tag{24.9}$$

where we have collected all the dimensional quantities into a single parameter

$$\mu \equiv \frac{\mathcal{N} d^2 \omega}{\varepsilon_0 \hbar c}$$

24.2 Propagation and Beer's Law

To find the dynamics of the field during propagation, we need to know u and v. According to the Bloch equations, we have

$$\frac{d}{dt}(u - iv) = i(\Delta + \dot{\phi})(u - iv) - iRw$$

which can be solved formally as

$$(u - iv)_t = -i \int_{-\infty}^{t} dt' \, e^{i\Delta(t-t')} \, e^{i[\phi(t) - \phi(t')]} \, R(t') w(t')$$

where the lower limit $t = -\infty$ is simply the time before the arrival of the light pulse, and we have assumed that initially, all the atoms are in the ground state.

We recall that in the presence of inhomogeneous broadening, we have to average over the distribution of detuning $g(\Delta)$. We will assume a broad and smooth distribution, implying a very rapid inhomogeneous decay. This will impose a short-memory Markov-like condition on the time evolution. In addition, we assume a weak excitation, i.e., the population remains largely in the ground state, hence $w(t) \approx -1$ at all t. With these assumptions, the averaged $u - iv$ becomes

$$\langle u - iv \rangle_t = iR(t) \int d\Delta \, g(\Delta) \int_{-\infty}^{t} dt' \, e^{i\Delta(t-t')} \tag{24.10}$$

The integral on the r.h.s. can be evaluated as (shift the time variable to $s = t - t'$)

$$\int d\Delta \, g(\Delta) \int_{-\infty}^{t} dt' \, e^{i\Delta(t-t')} = \int d\Delta \, g(\Delta) \int_{0}^{\infty} ds \, e^{i\Delta s} = \int d\Delta \, g(\Delta) \left(\pi\delta(\Delta) + i\frac{\mathcal{P}}{\Delta}\right) = \pi g(0) + i \int d\Delta \, g(\Delta) \frac{\mathcal{P}}{\Delta}$$

where $\mathcal{P}$ denotes the principal part of the integral.

Therefore we have

$$u = -R \int d\Delta \, g(\Delta) \frac{\mathcal{P}}{\Delta}, \quad v = -R\pi g(0)$$

and the propagation equations become

$$\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right) R(z,t) = -\frac{\pi\mu}{2} g(0) R(z,t) \tag{24.11}$$

$$\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right) \phi(z,t) = \frac{\mu}{2} \int d\Delta \, g(\Delta) \frac{\mathcal{P}}{\Delta} \tag{24.12}$$

The equation for R has the solution

$$R(z,t) = R(0, t - z/c)\, e^{-\alpha z/2}$$

with $\alpha = \pi\mu g(0)$ being the absorption coefficient, which indicates that the electric field maintains its pulse shape during propagation but the pulse amplitude is attenuated exponentially at the rate $\alpha/2$. Converting to pulse intensity which is proportional to the square of R, we have

$$I(z,t) = I(0, t - z/c)\, e^{-\alpha z}$$

This is known as the *Beer's Law*. The rate α is associated with the inhomogeneously broadened cross section as $\alpha = \mathcal{N}\sigma$, hence

$$\sigma = \frac{\pi\mu g(0)}{\mathcal{N}} = \frac{\pi d^2 \omega}{\varepsilon_0 \hbar c}\, g(0)$$

24.3 Area Theorem

Earlier we have shown that under the factorization ansatz

$$v(t;\Delta) = F(\Delta) v(t;0) \tag{24.13}$$

the pulse area satisfies a pendulum equation

$$\ddot{\Theta} - \frac{1}{\tau_p^2} \sin\Theta = 0 \tag{24.14}$$

Now we will derive the *Area Theorem* by returning to the propagation equation

$$\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right)\left[R(z,t) e^{-i\phi(z,t)}\right] = i\frac{\mu}{2}\langle u - iv\rangle\, e^{-i\phi(z,t)}$$

Next we will substitute $\langle u - iv\rangle$ into the propagation equation. The solution for $\langle u - iv\rangle$ is the same as in (25.7) except that we now retain the dynamics properties of w:

$$\langle u - iv\rangle = -iR(t) \int d\Delta\, g(\Delta) \int_{-\infty}^{t} dt'\, e^{i\Delta(t-t')}\, w(z,t;\Delta)$$

Eqs. (24.11) and (24.12) will be modified as

$$\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right) R(z,t) = \frac{\pi\mu}{2} g(0)\, w(z,t;0)\, R(z,t) \tag{24.15}$$

$$\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right) \phi(z,t) = -\frac{\mu}{2} \int d\Delta\, g(\Delta)\, \frac{\mathcal{P}}{\Delta}\, w(z,t;\Delta) \tag{24.16}$$

To obtain an equation for area Θ, we must now integrate both sides of (24.15) over time from $t = -\infty$ to $t = \infty$. Since $R(t \to \pm\infty) = 0$, the integral of the t-derivative on the l.h.s. gives zero contribution, so we have

$$\frac{\partial}{\partial z}\Theta = \frac{\pi\mu}{2} g(0) \int dt\, w(z,t;0)\, R(z,t) = \frac{\pi\mu}{2} g(0) \int d\Theta\, w(z,t;0)$$

Since $w(z,t;0) = -\cos\Theta$, we have

$$\frac{\partial}{\partial z}\Theta(z) = -\frac{\alpha}{2} \int d\Theta\, \cos\Theta = -\frac{\alpha}{2} \sin\Theta(z) \tag{24.17}$$

This is the Area Theorem derived by McCall and Hahn in 1967.

In the limit for weak field, i.e., for small pulse area, $\sin\Theta \approx \Theta$, then we have

$$\frac{\partial}{\partial z}\Theta(z) = -\frac{\alpha}{2}\Theta(z), \quad \text{or} \quad \Theta(z) = \Theta(0)\, e^{-\alpha z/2}$$

which is just the Beer's Law. Hence the Beer's Law is a special case the Area Theorem.

Area Theorem is remarkable in several ways. First it works for an arbitrary amount of inhomogeneous broadening. Second, it predicts multiple steady state solutions for the values $\Theta = n\pi$, where $n = 0, 1, 2, ...$ However only when n is even the solution is stable. This is shown in the figure below: as the pulse propagates, the area tends toward even multiples of π. Note that a stable area does not necessarily imply a stable pulse shape as $R(t)$ can change in time with Θ being fixed. Next we will see what kind of pulses also have a constant shape.

24.4 Optical Solitons and Self-Induced Transparency

We have seen that the factorization ansatz (24.13) of the OBEs permits a special solution. Now we want to put this into perspective in the context of the propagation equation and demonstrate that a hyperbolic secant 2π-pulse which is both area-stable and shape-stable is indeed a solution of the Maxwell-Bloch equations.

We will start with the propagation equation

$$\frac{\partial}{\partial t}\left(\frac{\partial}{\partial z}+\frac{1}{c}\frac{\partial}{\partial t}\right)\Theta=-\frac{\mu}{2}\langle F(\Delta)\rangle\,\sin\Theta \tag{24.18}$$

where we have used $R=\dot{\Theta}$ again. This is a nonlinear wave equation since $\sin\Theta$ is a nonlinear function of Θ.

To anticipate a shape-preserving solution, we will define a local time

$$\xi\equiv t-z/V$$

where V will be identified as the pulse velocity. In terms of ξ, we have

$$\frac{\partial}{\partial t}\rightarrow\frac{d}{d\xi},\quad \frac{\partial}{\partial z}\rightarrow-\frac{1}{V}\frac{d}{d\xi}$$

Hence we can rewrite (24.18) as

$$\frac{d^2}{d\xi^2}\Theta=\frac{1}{\tau_p^2}\,\sin\Theta \tag{24.19}$$

if we define τ_p as

$$\frac{\mu}{2}\langle F(\Delta)\rangle\,\tau_p^2=\frac{1}{V}-\frac{1}{c} \tag{24.20}$$

Equation (24.19) is identical to (24.14) and can therefore be solved in the same manner. The solution that fits our boundary conditions is

$$\Theta(\xi)=4\tan^{-1}\left(e^{\xi/\tau_p}\right)$$

or in terms of the Rabi frequency

$$R(t,z)=\frac{1}{\tau_p}\,\mathrm{sech}\,\frac{\xi}{\tau_p}=\frac{1}{\tau_p}\,\mathrm{sech}\,\frac{t-z/V}{\tau_p}$$

This 2π hyperbolic secant pulse is quite remarkable. It is travelling in an absorbing medium (remember that all the atoms are initially in the ground state, hence they can only *absorb* energy from the light), but managing to do so without loss and without changing its shape. In other words, the medium acts as if it were transparent. Since this unexpected transparency is produced by the self-consistent field-atom interaction, McCall and Hahn named the phenomenon *self-induced transparency* or SIT. Due to its shape-preserving feature, the hyperbolic secant pulse is an *optical soliton.*

The pulse width τ_p and pulse velocity V are closely related through (24.20). For short pulses, $V\rightarrow c$, but V can be very slow for long pulses. Using $F(\Delta)=1/[1+(\Delta\tau_p)^2]$, we have

$$\langle F(\Delta)\rangle=\int d\Delta\,\frac{g(\Delta)}{1+(\Delta\tau_p)^2}$$

When $\tau_p\gg 1/\delta\omega$ (where $\delta\omega$ is the inhomogeneous broadened linewidth), we have

$$\frac{1}{1+(\Delta\tau_p)^2}=\frac{1}{\tau_p}\,\frac{1/\tau_p}{\Delta^2+(1/\tau_p)^2}\approx\frac{\pi}{\tau_p}\delta(\Delta)$$

then we have

$$\langle F(\Delta)\rangle = \pi g(0)/\tau_p$$

from which it follows that

$$V = \frac{c}{1 + \alpha c\tau_p/2}$$

which can be much smaller than c if $\alpha c\tau_p$ is large.

24.5 SIT and Pulse Propagation

An obvious question one may ask is how can the pulse velocity significantly lower than c if the medium is "transparent". Of course, the medium is only transparent in terms of energy absorption. Since a 2π pulse turns the Bloch vector of an atom from ground state back to ground state, the atom can take no energy from it. This explains the lack of net absorption by the medium. But this process of turning the Bloch vector around must be done fully coherently as the atom absorbs and re-radiates all the energy back to the pulse. This cannot be done more quickly than the inhomogeneous relaxation time $1/\delta\omega$. That is why the pulse is significantly slowed down.

Another natural question that can be asked is: how can this magic pulse shape be achieved in practice and whether it is stable? The hint to this question is provided by the Area Theorem. The Area Theorem guarantees that the area 2π is stable and that *all* areas between π and 3π will evolve to 2π. If the pulse returns to area 2π, but not to the secant shape, it will have to change its shape (since only secant is a stable shape). In changing its shape, it probably has to change its area again, and so on. Numerical simulations show that this process converges rather quickly, i.e., the pulse adjusts itself automatically to the magic secant shape as illustrated in the figure below.

What if the input pulse has area $2n\pi$ with $n > 1$? A $2n\pi$ pulse is area-stable, but not shape-stable. As it propagates, the pulse will split into n 2π pulses with the magic hyperbolic secant shape.

Chapter 25

Rate Equations and Saturation Phenomena

In our discussion of two-level atoms interacting with a classical laser field, we have neglected the relaxation by assuming that the time of interest is much shorter than the relaxation time T_1 and T_2. However, this restriction does not limit the range of interesting and important resonant interaction phenomena. Here we want to study the incoherent resonance phenomena occurring over times that may be much longer the relaxation time.

Traditionally incoherent optical effects are described by simple detailed-balance *rate equations*. As we shall see, these rate equations are merely a special case of the coherent Bloch or Maxwell equations in the quasi-steady-state limit.

25.1 Rate Equations of the OBEs

The rate equations frequently assumed to describe a two-level atom interacting with a laser field are

$$\dot{n}_e = -R'(n_e - n_g) - n_e/T_1 \tag{25.1}$$

$$\dot{n}_g = R'(n_e - n_g) + n_e/T_1 = -\dot{n}_e \tag{25.2}$$

where $n_e = \mathcal{N}\sigma_{ee}$ ($n_g = \mathcal{N}\sigma_{gg}$) is the excited (ground) state population density, R expresses the rate of stimulated emission and absorption due to the applied field. The three terms on the r.h.s. of (25.1) and (25.2) can be intuitively understood as arising from stimulated emission, absorption and spontaneous emission.

Using $w = \sigma_{ee} - \sigma_{gg}$ and $1 = \sigma_{ee} + \sigma_{gg}$, we have

$$n_e = \frac{\mathcal{N}}{2}(1+w), \quad n_g = \frac{\mathcal{N}}{2}(1-w)$$

The two rate equations can be transformed to a single equation for the inversion w as

$$\dot{w} = -2R'w - \frac{w+1}{T_1} \tag{25.3}$$

The question of interest is: what connection can this equation for the inversion have with the one we obtained from the OBEs?

The OBEs including the relaxation terms are

$$\dot{u} = \Delta v - u/T_2 \tag{25.4}$$

$$\dot{v} = -\Delta u + Rw - v/T_2 \tag{25.5}$$

$$\dot{w} = -Rv - (1+w)/T_1 \tag{25.6}$$

If T_2 is very short ($T_2 \ll T_1$), then u and v will quickly reach the quasi-steady-state values which can be easily obtained from (25.4) and (25.5) as

$$u = \Delta R T_2^2 \mathcal{L}\, w, \quad v = R T_2\, \mathcal{L}\, w \tag{25.7}$$

where the Lorentzian factor $\mathcal{L}$ is given by $\mathcal{L} \equiv 1/[1 + (\Delta T_2)^2]$.

When v in the r.h.s. of (25.6) is replace its quasi-steady-state value, we have

$$\dot{w} = -\frac{\mathcal{L} I}{T_1} w - \frac{w+1}{T_1} \tag{25.8}$$

where $I = R^2 T_1 T_2$ is the "dimensionless intensity".

Compare (25.8) with (25.3), we immediate find that the rate equation is obtained in this quasi-steady-state limit which is valid for time $t \gg T_2$.

With initial value $w(0) = w_0$ and under the assumption that field intensity I is time-independent, Equation (25.8) can be solved as

$$w(t) = -\frac{1}{1+I\mathcal{L}} + \left(w_0 + \frac{1}{1+I\mathcal{L}} \right) e^{-\frac{1+I\mathcal{L}}{T_1} t}$$

The solution $w(t)$ is plotted in Fig. 25-1. Several features of $w(t)$ are obvious in the figure. The decay rate is influenced both by detuning and by the field strength through the parameter $(1 + I\mathcal{L})$, and can be substantially greater than $1/T_1$.

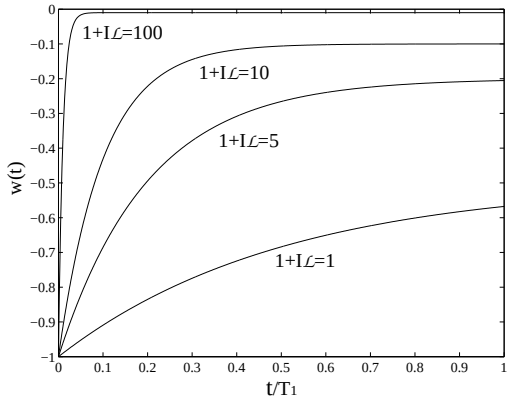

Figure 25-1: Inversion as a function of time.

For for sufficiently long time, w will also reach the steady-state value

$$w_{\rm ss} = -\frac{1}{1+I\mathcal{L}} = -\frac{1+(\Delta T_2)^2}{1+(\Delta T_2)^2+I} \tag{25.9}$$

which depends on the detuning and the intensity. This steady-state value is plotted in Fig. (25-2) as a function of detuning. As we can see, the width of $w_{\rm ss}$ increases with I. This phenomenon is called the *power broadening*.

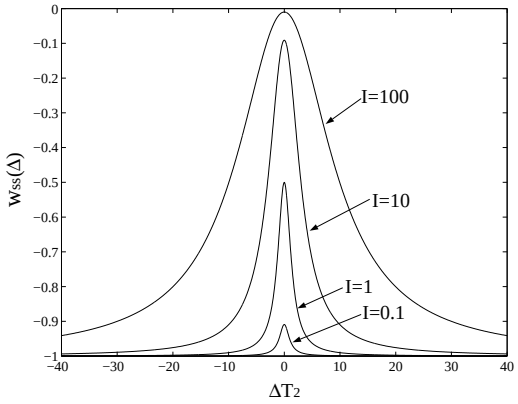

Figure 25-2: Steady-state inversion.

25.2 Rate Equations of the Maxwell Equations

In the general case, the field intensity I is of course not constant, but obeys an equation of motion of its own. This equation of motion is ultimately derived from the Maxwell equations, which is now coupled with the OBEs.

Intuitively, we can write down a rate equation for the field intensity as

$$\frac{\partial I}{\partial z} = \mathcal{N}\sigma I w = \sigma(n_e - n_g)I \tag{25.10}$$

where σ is the emission/absorption cross section, and the two terms on the r.h.s. represents field gain from stimulated emission and field loss from absorption. As in the previous case, we can similarly ask: how can this equation be derived from the Maxwell equation?

As we have shown, the field amplitude, or equivalently the Rabi frequency R satisfies the following propagation equation:

$$\left(\frac{\partial}{\partial z} + \frac{1}{c}\frac{\partial}{\partial t}\right) R = \frac{\mu}{2} v \tag{25.11}$$

For sufficiently long time, the field reaches steady state, then we can neglect the time derivative on the l.h.s. and replace v on the r.h.s. by its quasi-steady-state value in (25.7), we have

$$\frac{\partial}{\partial z} R = \frac{\mu}{2} R T_2 \mathcal{L} w \tag{25.12}$$

In order to get an equation for the intensity $I = R^2 T_1 T_2$, we multiply both sides of the equation by RT_1T_2, which yields

$$\frac{1}{2}\frac{\partial}{\partial z} I = \frac{\mu}{2} T_2 \mathcal{L} w I$$

This equation has the same form as (25.10) with $\mathcal{N}\sigma = \mu T_2 \mathcal{L}$.

If significant inhomogeneous broadening exists, then we need to average $\mathcal{L}w$ at the r.h.s. of (25.12) over the inhomogeneous lineshape as

$$\langle \mathcal{L} w \rangle = \int d\Delta'\, g(\Delta') \frac{w(z;\Delta')}{1+(\Delta' T_2)^2} = \frac{\pi g(0)}{T_2} w(z;0)$$

where we have approximated the Lorentzian factor $\mathcal{L}$ by a δ-function:

$$\mathcal{L} = \frac{1}{1+(\Delta T_2)^2} = \frac{1}{T_2}\,\frac{1/T_2}{\Delta^2 + (1/T_2)^2} \to \frac{\pi}{T_2}\,\delta(\Delta)$$

which is valid when $\delta\omega \gg 1/T_2$. Again we can reduce Eq. (25.12) to the form of (25.10) with $\mathcal{N}\sigma = \mu\pi g(0) = \alpha$.

25.3 Saturation Spectroscopy

In the situation where a weak probe field interacting with an absorption medium composed of an ensemble of ground state atoms with large inhomogeneous broadening, the absorption spectrum of the probe field is governed by the inhomogeneous distribution function $g(\Delta)$, whose width is normally much wider than the homogenous linewidth (or natural linewidth) $1/T_1$. To obtain the information on the homogeneous linewidth, saturation spectroscopy is usually employed. Let us consider how this works.

The absorption of the probe is governed by Eq. (25.12). For saturation spectroscopy, the weak probe field is applied along with a strong pump field. The latter drives the atom to steady-state with

$$w_{\rm ss} = -\frac{1}{1+\mathcal{L}I}$$

Here the parameters are related to the pump field.

From the viewpoint of the probe field, the medium has been modified. The modification can be characterized by the *saturated* inhomogeneous lineshape function

$$g_{\rm sat}(\Delta') = -w_{\rm ss} g(\Delta') = \frac{g(\Delta')}{1+\mathcal{L}I}$$

which is plotted in Fig. 25-3. As one can see, $g_{\rm sat}$ has a "hole" whose width is given by the width of $w_{\rm ss}$, which in term yields the value of $1/T_2$. Of course, it is not that there are less atoms in the hole region, only that these atoms are busy interacting with the pump field (i.e., "burned" by the pump), hence do not do their fair share of work in absorbing the probe field. Such a phenomenon is referred to as the *hole burning*.

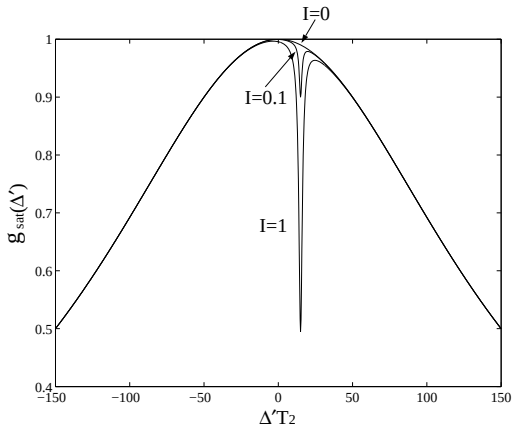

Figure 25-3: Saturated inhomogeneous lineshape, exhibiting hole burning.

Chapter 26

Coherent Effects in Three-Level Atom

Quantum coherence and correlations in quantum optics lead to many interesting and unexpected consequences. Some of the coherence phenomena we have encountered include quantum beats and photon echo. Here we will discuss a few examples in three-level Λ atom where coherence plays vital roles.

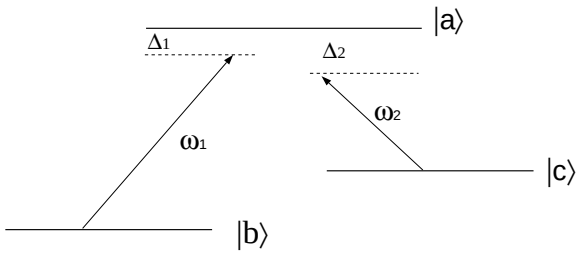

Figure 26-1: Level scheme of a Λ atom.

26.1 Coherent Population Trapping and Dark State

The energy levels of a Λ atom are illustrated in the figure above. Two laser fields with frequencies ω_1 and ω_2, respectively, drives the two dipole transitions $|a\rangle \leftrightarrow |b\rangle$ and $|a\rangle \leftrightarrow |c\rangle$. The semi-classical Hamiltonian contains the bare atomic part (take the energy reference at the level $|b\rangle$), H_A, and the interaction part, $H_{\rm int}$, which are given by

$$H_A = \hbar\omega_a|a\rangle\langle a| + \hbar\omega_c|c\rangle\langle c| \tag{26.1}$$

$$H_{\rm int} = \frac{\hbar}{2}\Omega_1 e^{-i\omega_1 t}\,|a\rangle\langle b| + \frac{\hbar}{2}\Omega_2 e^{-i\omega_2 t}\,|a\rangle\langle c| + h.c. \tag{26.2}$$

The state vector has the general form

$$|\psi(t)\rangle = c_a(t)e^{-i\omega_a t}\,|a\rangle + c_b(t)|b\rangle + c_c(t)e^{-i\omega_c t}\,|c\rangle$$

from which we can derive the equations of motion for the amplitudes as

$$i\dot{c}_a = \frac{1}{2}\Omega_1 c_b e^{-i\Delta_1 t} + \frac{1}{2}\Omega_2 c_c e^{-i\Delta_2 t} \tag{26.3}$$

$$i\dot{c}_b = \frac{1}{2}\Omega_1^* c_a e^{i\Delta_1 t} \tag{26.4}$$

$$i\dot{c}_c = \frac{1}{2}\Omega_2^* c_a e^{i\Delta_2 t} \tag{26.5}$$

where $\Delta_1 = \omega_1 - \omega_a$ and $\Delta_2 = \omega_2 - (\omega_a - \omega_c)$ are the one-photon detunings for the two dipole transitions, respectively.

Now let us consider the special situation $\Delta_1 = \Delta_2 = \Delta$ and try to solve the amplitudes equations. Take one more time derivative on both sides of (26.3) leads us to

$$\begin{aligned}\ddot{c}_a &= -\frac{1}{2}\Omega_1(i\dot{c}_b)e^{-i\Delta t} - \frac{1}{2}\Omega_1\Delta c_b\, e^{-i\Delta t} - \frac{1}{2}\Omega_2(i\dot{c}_c)e^{-i\Delta t} - \frac{1}{2}\Omega_2\Delta c_c\, e^{-i\Delta t} \\ &= -\frac{1}{4}\left(|\Omega_1|^2 + |\Omega_2|^2\right)c_a - i\Delta\dot{c}_a\end{aligned}$$

which can be solve as

$$c_a(t) = Ae^{\omega_1 t} + Be^{\omega_2 t}$$

where $\omega_{1,2}$ are the solution to the characteristic equation

$$\ddot{x} + i\Delta x + \frac{1}{4}\left(|\Omega_1|^2 + |\Omega_2|^2\right) = 0$$

i.e.,

$$\omega_{1,2} = \frac{i}{2}\left(-\Delta \pm \sqrt{\Delta^2 + |\Omega_1|^2 + \Omega_2|^2}\right)$$

and the constant coefficients A and B are fixed by the initial condition as

$$c_a(0) = A + B, \quad \dot{c}_a(0) = -\frac{i}{2}\Delta(A+B) + i\sqrt{\Delta^2 + |\Omega_1|^2 + \Omega_2|^2}(A - B) = -\frac{i}{2}[\Omega_1 c_b(0) + \Omega_2 c_c(0)]$$

Now one can see that if the atom is initially prepared in a coherent superposition state with

$$c_a(0) = 0, \quad c_b(0) = \frac{\Omega_2}{\sqrt{|\Omega_1|^2 + |\Omega_2|^2}}, \quad c_c(0) = -\frac{\Omega_1}{\sqrt{|\Omega_1|^2 + |\Omega_2|^2}} \tag{26.6}$$

then we have $A = B = 0$, hence $c_a(t)$ remains at zero all the time, and according to (26.4) and (26.5), $c_b(t)$ and $c_c(t)$ also maintain their initial values since their time derivatives vanish.

Further inspection shows that with the initial amplitudes given in (26.6), i.e., with the initial state vector

$$|\psi(0)\rangle = \frac{1}{\sqrt{|\Omega_1|^2 + |\Omega_2|^2}}\left(\Omega_2|b\rangle - \Omega_1 e^{-i\omega_c t}|c\rangle\right) \tag{26.7}$$

we have

$$H_{\text{int}}|\psi(0)\rangle = 0$$

In other words, this state is NOT coupled by the laser light. State (26.7) is therefore call the *dark state*. Once the atom is prepared in the dark state, it remains there and the population is locked to the ground states of the Λ atom. For this reason, the state is also called *coherent population trapping* state, or CPT state. Note that dark state only exists when $\Delta_1 = \Delta_2$, i.e., at two-photon resonance.

CPT can be understood as an *interference* phenomenon. The same final state $|a\rangle$ can be reached via two routes: $|b\rangle \rightarrow |a\rangle$ or $|c\rangle \rightarrow |a\rangle$. Two two routes thus interfere with each other. At two-photon resonance, the absorption amplitude cancels each other exactly, resulting in the dark state.

CPT state serves as the basis for many other applications such as adiabatic population transfer through STIRAP, lasing without inversion (LWI), electro-magnetically induced transparancy (EIT), etc.

26.2 STIRAP

STIRAP stands for *stimulated rapid adiabatic passage*. The technique of STIRAP is used to *coherently* transfer populations from one state to another state.

Consider the three-level Λ atom. Initially all the populations are in state $|b\rangle$ and we want to transfer it to $|c\rangle$. We notice from (26.7) that

$$\frac{\Omega_1}{\Omega_2} \rightarrow 0, |b\rangle \rightarrow \text{ dark state}, \qquad \frac{\Omega_2}{\Omega_1} \rightarrow 0, |c\rangle \rightarrow \text{ dark state}$$

As the dark state is an eigenstate of the total Hamiltonian, according to the adiabatic theorem, when the Hamiltonian changes in time slowly enough, the dark state will follow the time-dependent Hamiltonian adiabatically. Hence all we need to do is to adiabatically change the ratio Ω_1/Ω_2 from 0 to ∞, then the population will be transferred from $|b\rangle$ to $|c\rangle$. During this whole process, the excited state $|a\rangle$ is never populated, therefore this process is not subject to decoherence due to the spontaneous emission from the excited state.

To make STIRAP work, we have $\Omega_1/\Omega_2 = 0$ initially. In other words, the laser field driving the $|a\rangle \leftrightarrow |c\rangle$ transition (note that $|c\rangle$ is unoccupied initially) needs to be turned on first. This sequence may seem a little counterintuitive at first sight. That is why it is sometime referred as a counterintuitive pulse sequence.

26.3 LWI

According to the conventional wisdom, lasing requires population inversion. Otherwise, the light field will experience net loss (absorption). However, taking advantage of the existence of the dard state, one can achieve lasing even though there is less population in the excited level $|a\rangle$ than that in $|b\rangle$ and $|c\rangle$. The reason is simple, if the populations in the two lower levels are trapped in the dark state, they cannot absorb the light, so from the lasing point of view, these populations do not exist.

26.4 EIT

The strange interactions between light and matter make many phenomena appear counter-intuitive. EIT is one of the examples. It says that propagate one laser beam through a medium and it will get absorbed; propagate two laser beams through the same medium and neither will be absorbed. A quite literal trick of the light turns an opaque media into a transparent one. Here let us demystify EIT using CPT. Qualitatively, the interactions of the two beams (conventionally called the pump and the probe) with the atoms pumps

the latter into the dark state or CPT state. Once the atoms are in the CPT state, no light absorption can take place.

To put this into more rigorous form, let us use the OBEs. We choose OBEs here over the amplitude equations as in Sec.I because we want to incorporate damping arising from the spontaneous emission of the excited state $|a\rangle$.

The Hamiltonian is given in Sec.I. The OBEs can be derived as

$$\begin{aligned}
i\dot{\sigma}_{bb} &= i\gamma_b\sigma_{aa} - \frac{\Omega_1}{2}e^{-i\omega_1 t}\,\sigma_{ab} + \frac{\Omega_1^*}{2}e^{i\omega_1 t}\,\sigma_{ba} \\
i\dot{\sigma}_{cc} &= i\gamma_c\sigma_{aa} - \frac{\Omega_2}{2}e^{-i\omega_2 t}\,\sigma_{ac} + \frac{\Omega_2^*}{2}e^{i\omega_2 t}\,\sigma_{ca} \\
i\dot{\sigma}_{ab} &= -i\frac{\gamma_b+\gamma_c}{2}\sigma_{ab} - \omega_a\sigma_{ab} - \frac{\Omega_2^*}{2}e^{i\omega_2 t}\,\sigma_{cb} + \frac{\Omega_1^*}{2}e^{i\omega_1 t}\,(\sigma_{aa}-\sigma_{bb}) \\
i\dot{\sigma}_{ac} &= -i\frac{\gamma_b+\gamma_c}{2}\sigma_{ac} + \omega_c\sigma_{ab} - \frac{\Omega_1^*}{2}e^{i\omega_1 t}\,\sigma_{bc} + \frac{\Omega_2^*}{2}e^{i\omega_2 t}\,(\sigma_{aa}-\sigma_{cc}) \\
i\dot{\sigma}_{bc} &= \omega_c\sigma_{bc} - \frac{\Omega_1}{2}e^{-i\omega_1 t}\,\sigma_{ac} + \frac{\Omega_2^*}{2}e^{i\omega_2 t}\,\sigma_{ba}
\end{aligned}$$

Here γ_b and γ_c are the population decay rate from the excited state $|a\rangle$ to $|b\rangle$ and $|c\rangle$, respectively.

To get rid of the fast oscillating terms, let us define a set of slowly-varying quantities as

$$\begin{aligned}
\tilde{\sigma}_{ab} &= e^{-i\omega_1 t}\,\sigma_{ab}, \quad \tilde{\sigma}_{ba} = e^{i\omega_1 t}\,\sigma_{ba} \\
\tilde{\sigma}_{ac} &= e^{-i\omega_2 t}\,\sigma_{ac}, \quad \tilde{\sigma}_{ca} = e^{i\omega_2 t}\,\sigma_{ca} \\
\tilde{\sigma}_{bc} &= e^{(i\omega_1-\omega_2)t}\,\sigma_{bc}, \quad \tilde{\sigma}_{cb} = e^{-i(\omega_1-\omega_2)t}\,\sigma_{cb} \\
\tilde{\sigma}_{ii} &= \sigma_{ii}, \;\; i = a,b,c
\end{aligned}$$

The OBEs in terms of these slowly-varying variables are

$$\begin{aligned}
i\dot{\tilde{\sigma}}_{bb} &= i\gamma_b\tilde{\sigma}_{aa} - \frac{\Omega_1}{2}\tilde{\sigma}_{ab} + \frac{\Omega_1^*}{2}\tilde{\sigma}_{ba} \\
i\dot{\tilde{\sigma}}_{cc} &= i\gamma_c\tilde{\sigma}_{aa} - \frac{\Omega_2}{2}\tilde{\sigma}_{ac} + \frac{\Omega_2^*}{2}\tilde{\sigma}_{ca} \\
i\dot{\tilde{\sigma}}_{ab} &= -i\frac{\gamma_b+\gamma_c}{2}\tilde{\sigma}_{ab} + \Delta_1\tilde{\sigma}_{ab} - \frac{\Omega_2^*}{2}\tilde{\sigma}_{cb} + \frac{\Omega_1^*}{2}(\tilde{\sigma}_{aa}-\tilde{\sigma}_{bb}) \\
i\dot{\tilde{\sigma}}_{ac} &= -i\frac{\gamma_b+\gamma_c}{2}\tilde{\sigma}_{ac} + \Delta_2\tilde{\sigma}_{ac} - \frac{\Omega_1^*}{2}\tilde{\sigma}_{bc} + \frac{\Omega_2^*}{2}(\tilde{\sigma}_{aa}-\tilde{\sigma}_{cc}) \\
i\dot{\tilde{\sigma}}_{bc} &= (\Delta_2-\Delta_1)\tilde{\sigma}_{bc} - \frac{\Omega_1}{2}\tilde{\sigma}_{ac} + \frac{\Omega_2^*}{2}\tilde{\sigma}_{ba}
\end{aligned}$$

We are interested in the steady-state solutions which can be obtained by taking the time-derivative to be zero. Under the condition of the two-photon resonance $\Delta_1 = \Delta_2$, the steady state solutions are particularly simple:

$$\sigma_{bb} = \frac{|\Omega_2|^2}{|\Omega_1|^2+|\Omega_2|^2}, \quad \sigma_{cc} = \frac{|\Omega_1|^2}{|\Omega_1|^2+|\Omega_2|^2}, \quad \tilde{\sigma}_{bc} = \tilde{\sigma}_{cb}^* = -\frac{\Omega_1\Omega_2^*}{|\Omega_1|^2+|\Omega_2|^2}$$

and all the other σ_{ij}'s vanish. This can be identified as the CPT solution we found earlier.

When $\Delta_1 \neq \Delta_2$, analytical expressions of the steady-state solution can still be found (with the aid of Mathematica, for example), but are in general rather complicated. Simplifications, however, can be found under the following situation. Suppose the atoms are initially in state $|b\rangle$. The $|a\rangle \leftrightarrow |c\rangle$ is driven by a strong resonant pump field, while $|a\rangle \leftrightarrow |b\rangle$ is driven by a weak probe field, i.e., $|\Omega_2| \gg |\Omega_1$ and $\Delta_2 = 0$. Then the

population will roughly stay in $|b\rangle$. We are particularly interested in how the system responds when Δ_1 is varied.

Take $\sigma_{bb} = 1$ and $\sigma_{aa} = \sigma_{cc} = 0$, the steady-state solution of the OBEs can be easily found as

$$\tilde{\sigma}_{ba} = \frac{\Omega_1}{2\Delta_1 + i\gamma - \frac{|\Omega_2|^2}{2\Delta_1 + i|\Omega_1|^2/\gamma}} \approx \frac{2\Delta_1\Omega_1}{(4\Delta_1^2 - |\Omega_2|^2) + i2\Delta_1\gamma} \tag{26.8}$$

where $\gamma \equiv \gamma_b + \gamma_c$ and we have neglected the term $|\Omega_1|^2/\gamma$ which is supposed to be small.

Neglect the inhomogeneous broadening, the slowly-varying amplitude of the polarization density for the probe field is given by

$$\mathcal{P}_{ba} = \mathcal{N} d_{ba} \tilde{\sigma}_{ba} = \varepsilon_0 \chi \mathcal{E}_1$$

where d_{ba} is the dipole moment for the $|a\rangle \leftrightarrow |b\rangle$ transition, $\chi = \chi' + i\chi''$ is the so-called *susceptibility*. Using $\Omega_1 = -2d_{ba}\mathcal{E}_1/\hbar$, we have

$$\chi' = \frac{4\mathcal{N}d_{ba}^2}{\varepsilon_0\hbar} \frac{\Delta_1(|\Omega_2|^2 - 4\Delta_1^2)}{(4\Delta_1^2 - |\Omega_2|^2)^2 + 4\Delta_1^2\gamma^2} \tag{26.9}$$

$$\chi'' = \frac{4\mathcal{N}d_{ba}^2}{\varepsilon_0\hbar} \frac{2\Delta_1^2\gamma}{(4\Delta_1^2 - |\Omega_2|^2)^2 + 4\Delta_1^2\gamma^2} \tag{26.10}$$

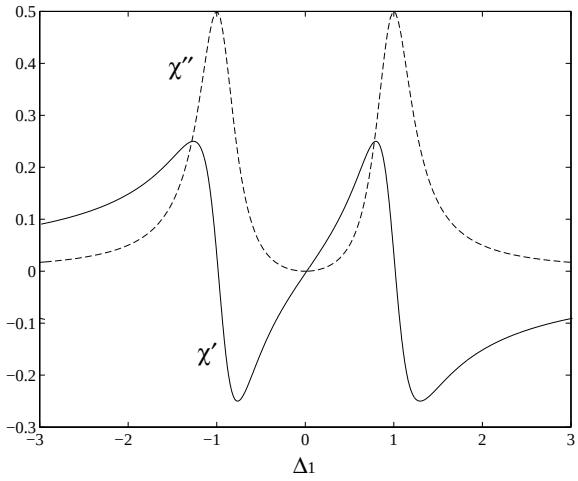

Figure 26-2: Real and imaginary part of the susceptibility.

The real and imaginary part of the susceptibility as a function of the probe detuning Δ_1 is plotted in the figure above. It can be seen that, at $\Delta_1 = 0$ (i.e., two-photon resonance since $\Delta_2 = 0$), both χ' and χ'' are zero. The real part χ' is related to the probe dispersion, while the imaginary part χ'' is related to the probe absorption. Hence at $\Delta_1 = 0$, there is neither absorption nor dispersion. In other words, the *medium becomes transparent*.

As we have mentioned, the origin of EIT can be understood in terms of the dark state. The atoms are initially prepared in state $|b\rangle$, but quickly get pumped into the dark state by the combined action of the strong pump and weak probe, as well as the spontaneous emission.

In the absence of the pump field, i.e., $\Omega_2 = 0$, Eqs. (26.9) and (26.10) reduce to

$$\chi' = -\frac{4\mathcal{N}d_{ba}^2}{\varepsilon_0\hbar} \frac{\Delta_1}{4\Delta_1^2 + \gamma^2}, \quad \chi'' = \frac{4\mathcal{N}d_{ba}^2}{\varepsilon_0\hbar} \frac{\gamma/2}{4\Delta_1^2 + \gamma^2}$$

which yields the familiar dispersive and absorptive curves for a two-level atom.

26.5 EIT and Slow Light

Quantum coherence allows us to manipulate the speed of light. To see how this can happen, let us again focus on the EIT system. The *refractive index* is related to the susceptibility as

$$n(\omega) = n'(\omega) + in''(\omega) = \sqrt{1 + \chi(\omega)}$$

where the argument ω emphasizes that both n and χ are frequency-dependent. Near $\Delta_1 = 0$, we have

$$\chi' \approx \chi'' \approx 0$$

Therefore we have

$$n' \approx 1 + \chi'/2 \approx 1, \quad n'' \approx \chi''/2 \approx 0$$

According to these relationships, it may seem that the medium behaves like a vacuum. But this is far from the truth. For example, the *group velocity* of the light can be significantly smaller than c.

Too see this, we need to first introduce the group velocity v_g and that of the phase velocity v_p. These concepts can be most easily understood in the following situation. An ideal plane propagating along z can be written as (take the amplitude to be unity)

$$A(z,t) = \cos(kz - \omega t)$$

where k and ω are the wave number and frequency, respectively. It is easy to see that $A(x,t)$ is the solution of the 1D wave equation

$$\left(\frac{\partial^2}{\partial z^2} - \frac{1}{v_p^2}\frac{\partial^2}{\partial t^2}\right) A(z,t) = 0$$

where $v_p = \omega/k$ is the speed at which the shape of the wave is travelling, i.e., the speed at which any fixed phase of the cycle is displaced, and hence is called the *phase velocity*.

Now consider the superposition of two such plane waves. The two plane waves have wave number $k_1 = k + \delta k$ and $k_2 = k - \delta k$, and frequency $\omega_1 = \omega + \delta\omega$ and $\omega_2 = \omega - \delta\omega$, respectively. Their superposition can be written as

$$\cos(k_1 z - \omega_1 t) + \cos(k_2 z - \omega_2 t) = 2\cos(kz - \omega t)\,\cos(\delta k z - \delta\omega t)$$

This can be somewhat loosely interpreted as a simple sinusoidal wave with the angular velocity (frequency) ω, the wave number k, and the modulated amplitude $2\cos(\delta kz - \delta\omega t)$. In other words, the amplitude of the wave is itself a wave, and the phase velocity of this modulation wave is $v = \delta\omega/\delta k$. Since each amplitude wave contains a group of internal waves (in this example, it contains two waves), this speed is usually called the *group velocity*.

Now we can define the phase and group velocities as

$$v_p = \frac{\omega}{k}, \quad v_g = \frac{d\omega}{dk}$$

where ω and k are the frequency and wave number of the light field, respectively. In vacuum, we have $v_g = v_p = c$. The phase velocity inside a medium is modified as

$$v_p = \omega/k = c/n', \quad \text{or} \quad n'(\omega) = ck/\omega$$

Therefore we have

$$\frac{dn'}{d\omega} = -\frac{ck}{\omega^2} + \frac{c}{\omega}\frac{dk}{d\omega} = \frac{c}{\omega}\left(\frac{1}{v_g} - \frac{1}{v_p}\right)$$

which yields

$$v_g = \frac{1}{\frac{1}{v_p} + \frac{\omega}{c}\frac{dn'}{d\omega}} = \frac{c}{n' + \omega\frac{dn'}{d\omega}}$$

From this we see that in order to make v_g significantly different from v_p, strong dispersion (i.e., large $|dn'/d\omega|$) is required. Many types of media exhibit strong dispersion. However, in many of these media, strong dispersion is associated with strong absorption. In EIT material, on the other hand, strong dispersive region coincides with weak absorptive region.

Near $\Delta_1 = 0$, $n' \approx 1 + \chi'/2$. Hence

$$\frac{dn'}{d\omega} = \frac{1}{2}\frac{d\chi'}{d\omega} = \frac{2\mathcal{N}d_{ba}^2}{\varepsilon_0\hbar}\frac{1}{|\Omega_2|^2}$$

and $\omega\frac{dn'}{d\omega}$ can be significantly larger than 1, under which condition, we have

$$v_g = \frac{c}{\omega\frac{dn'}{d\omega}} = \frac{\varepsilon_0\hbar|\Omega_2|^2}{2\mathcal{N}d_{ba}^2}\,c$$

which decreases as the intensity of pump field decreases. This provides a knob to control v_g.

In the experiment reported in [Nature **397**, 594 (1999)], group velocity as small as 17m/s was detected. This is more than 7 orders of magnitude smaller than c!

Dispersion such that $dn'/d\omega > 0$ is called normal dispersion, while those with $dn'/d\omega < 0$ is called anomalous dispersion. In an anomalously dispersive medium, v_g can be larger than c, or may even become negative. So the light may appear to exit the medium *before* it arrives. These superluminal light pulses are, however, NOT at odds with special relativity or causality. They result from interference between different frequency components of the light fields inside the media.